

AXIS-PARALLEL SQUARE PACKING WITH AN EXACT FORMULA FOR OPTIMAL PACKING OF SEVEN SQUARES

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ABSTRACT. How large a square is needed to hold axis-parallel squares of prescribed side lengths? We give exact formulas for the optimal side through seven squares, with explicit constructions and geometric lower-bound proofs. For every $n \geq 6$, the optimum for n squares is a sum of at most $n - 3$ of the prescribed side lengths. This bound is sharp on an open set of side lengths and yields the optimal Euclidean Lipschitz constant $\sqrt{n - 3}$ of the optimum as a function of the n side lengths.

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1. INTRODUCTION

For every sequence $s = (s_i)_{i=0}^\infty$ of real numbers with $s_0 \geq s_1 \geq \dots \geq 0$, let $L^*(s)$ denote the smallest side length of an axis-parallel square that contains axis-parallel squares of side lengths $s_0, s_1, \dots$ with pairwise disjoint interiors. Rotations are not allowed and sides of length zero signify absent squares. By monotonicity and additivity of area and by a theorem of Moon and Moser [MM67], $L^*(s)$ is finite exactly when $s \in \ell_2$, and $\|s\|_2 \leq L^*(s) \leq \sqrt{2} \|s\|_2$. Hence L^* is defined on the cone $\ell_2^\downarrow$ of nonincreasing nonnegative sequences in ℓ_2 . We identify a finite nonincreasing nonnegative tuple with the sequence in $\ell_2^\downarrow$ obtained by appending zeros. For all $n \geq 1$ and $s \in \ell_2^\downarrow$, we write

$$L_n^*(s) = L^*(s_0, \dots, s_{n-1}, 0, 0, \dots)$$

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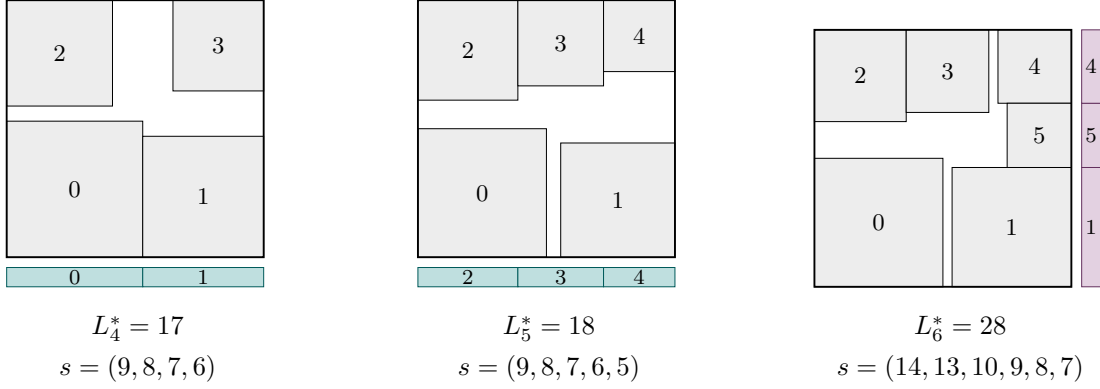


FIGURE 1.1. Optimal packings of four, five, and six squares. In the second and third panels, $L_5^* = L_4^* + 1$ and $L_6^* = L_5^* + 1$. A teal bar records the horizontal projections of a chain whose side sum is the side length of the square container; a purple bar on the right similarly records a vertical chain.

for the optimum for the n largest squares. Section 2 shows that L^* is continuous and L_n^* is piecewise linear for each n .

1.1. Exact formulas. The exact formulas through seven squares are as follows.

Theorem A. *For every $s \in \ell_2^\downarrow$, we have $L_1^*(s) = s_0$ and $L_n^*(s) = s_0 + s_1$ for $2 \leq n \leq 4$. For five and six squares,*

$$(1.1) \quad L_5^*(s) = \max(s_0 + s_1, s_2 + s_3 + s_4),$$

$$(1.2) \quad L_6^*(s) = \max(s_0 + s_1, s_2 + s_3 + s_4, s_1 + s_4 + s_5).$$

For seven squares,

$$(1.3) \quad L_7^*(s) = \max\left(L_6^*(s), \min\left\{\max(s_3 + s_4 + s_5 + s_6, s_1 + s_2 + s_5), \max(s_0 + s_5 + s_6, s_1 + s_3 + s_6), s_1 + s_4 + s_5 + s_6\right\}\right).$$

A horizontal chain is a family of squares whose horizontal projections have disjoint interiors; vertical chains are defined analogously. The side sum of a collection of squares is the sum of their side lengths. In any packing of axis-parallel squares, the side sum of a chain is at most the container side. The formulas for at most four squares are elementary. The five-square formula (1.1) is implicit in work of Lan, Dósa, Han, Zhou, and Benkő [LDH⁺13] on packing as many squares as possible into a unit square. In proving that a greedy shelf algorithm is optimal whenever at most five squares fit [LDH⁺13, Theorem 1], they show that five squares fit into a unit square if and only if the two largest sides sum to at most one and the three smallest sides sum to at most one; the second panel of Figure 1.1 illustrates this. We include a short proof of (1.1), since its chain argument is the starting point for six and seven squares.

The growth of complexity in the formula for L_n^* from $n = 6$ to $n = 7$ is reflected in the minimum size of a strategy family attaining the optimum for every nonincreasing side

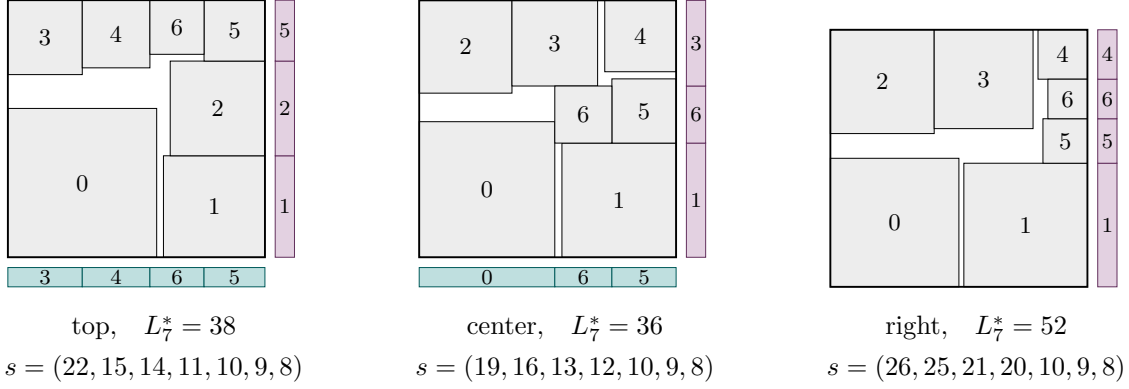


FIGURE 1.2. The “top”, “center”, and “right” packing strategies for seven squares at side vectors where the indicated strategy cost is uniquely optimal among the three strategies. The projection bars record the chains that determine the container side. Each example has $L_7^*(s) = L_6^*(s) + 1$. Labels and colors are as in Figure 1.1.

vector, which rises from one for six squares to three for seven squares; see Figure 1.2. In fact, L_7^* cannot be expressed as the minimum of two convex functions. In a companion paper [Bad26], we prove that L_8^* is the maximum of L_7^* and a minimum of twenty-three maxima of side sums. The proof there uses the methods developed here. At this time, a formula for L_9^* is unknown.

1.2. How many sides are needed to express the optimum? Section 2 shows that $L_n^*(s)$ is a piecewise linear function of $(s_0, \dots, s_{n-1})$ whose values are given by a sum of some of the side lengths. The next theorem identifies the largest possible minimum number of sides needed to express the optimum as a subset sum.

Theorem B. *For every $n \geq 6$ and every $s \in \ell_2^\downarrow$, there is a set $I \subseteq \{0, \dots, n-1\}$ with $|I| \leq n-3$ and $L_n^*(s) = \sum_{i \in I} s_i$. This bound is sharp: there is a nonempty open set $U \subseteq \{(s_0, \dots, s_{n-1}) \in \mathbb{R}^n : s_0 > s_1 > \dots > s_{n-1} > 0\}$ such that*

$$(1.4) \quad L_n^*(s) = s_1 + \sum_{i=4}^{n-1} s_i \quad \text{for all } s \in \ell_2^\downarrow \text{ with } (s_0, \dots, s_{n-1}) \in U,$$

and no subset sum with fewer than $n-3$ entries equals this value. Moreover, $L_n^*(s)$ is given by a finite minimum-of-maxima formula in $s_0, \dots, s_{n-1}$ involving only subset sums of cardinality at most $n-3$, and

$$(1.5) \quad |L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3} \|s - t\|_2 \quad \text{for all } s, t \in \ell_2^\downarrow,$$

where the constant $\sqrt{n-3}$ cannot be decreased.

1.3. Motivation and related packing problems. The author’s interest in L^* arose from joint work with Raanan Schul on Lipschitz parameterizations and rectifiable doubling measures [BS25]. In these constructions a tree organizes successive pieces of a target set,

and squares assigned to the children of a vertex must fit into a square assigned to their parent. The supremum metric is natural on the parameter plane because its balls are axis-parallel squares. Thus, $L_n^*(s)$ is the exact local cost of the packing operation. Replacing an upper estimate by this optimal cost avoids a loss at a step that is repeated throughout a multiscale construction. One useful estimate from [BS25] is

$$L_n^*(s) \leq \sum_{j^2 < n} s_{j^2} = s_0 + s_1 + s_4 + s_9 + \cdots.$$

It is obtained by successive L-shaped layers and gives $\lceil \sqrt{n} \rceil s$ for n equal sides of length s . The present paper studies the optimal function independently of those applications.

Classical packing theorems often give a guarantee in terms of total area, or optimize the area of a bounding rectangle. The theorem of Moon and Moser [MM67], and its extensions by Meir and Moser [MM68], are fundamental examples; see also Hougardy [Hou11] for packing squares of prescribed total area into a rectangle. For a fixed finite side vector, minimizing the side of a square container is of course equivalent to maximizing the occupied fraction $\sum_i s_i^2 / L^2$. Our aim is not a universal or asymptotic density estimate, but the exact dependence of the optimum on every prescribed side. Lower bounds for packing squares and rectangles into squares are analyzed by Caprara, Lodi, Martello, and Monaci [CLMM06].

Another well-known problem asks for the smallest square containing n *unit* squares with arbitrary rotations; Friedman [Fri09] surveys that problem and settles several of its small cases. With fixed orientations, equal sides are elementary: $L_n^*(1, \dots, 1) = \lceil \sqrt{n} \rceil$. Indeed, a grid gives the upper bound, and in a packing in $[0, L]^2$ the integer parts of the lower left corners are distinct points of $\{0, \dots, \lfloor L \rfloor - 1\}^2$. With free orientations, five unit squares already fit into a square of side $2 + 1/\sqrt{2} < 3$, and that value is optimal [Fri09].

For rectangles of fixed orientations, Murata, Fujiyoshi, Nakatake, and Kajitani [MFNK96] give a finite sequence-pair encoding and construct placements by longest paths in two constraint graphs. Their result applies to objectives nondecreasing in width and height, hence also to their maximum. There are $(n!)^2$ sequence-pairs. Silvanus and Vygen [SV20] represent a placement of n rectangles by assigning to each pair one of the relations west, south, east, or north, and call a set of such representations complete if it represents every placement without overlaps. They proved that the least size of a complete set lies between

$$\Omega\left(\frac{n!}{n^4} (4 + 2\sqrt{2})^n\right) \quad \text{and} \quad O\left(\frac{n!}{n^6} \left(\frac{11 + 5\sqrt{5}}{2}\right)^n\right).$$

In this paper, a complete family of strategies (Definition 2.3) need only attain $L_n^*(s)$ for every s , rather than represent every placement. The upper bound of Silvanus and Vygen applies to such families, but their lower bounds do not. Indeed, the least sizes of complete families for six and seven squares are 1 and 3 (Theorem 6.2), and for eight squares the least size is twenty-three [Bad26]. Subsection 2.2 gives the partial-order formulation suited to our arguments.

When n is part of the input, deciding whether prescribed squares fit orthogonally into a prescribed square is strongly NP-complete [LTW⁺90]. For each fixed n , Proposition 2.5 gives a finite formula for L_n^* ; the questions are its size and structure.

The rest of the paper is organized as follows. Section 2 develops general properties of L^* and of finite packing strategies. Section 3 introduces essential terminology used in proofs throughout the paper and proves Theorem A for up to six squares. Section 4 proves Theorem B. In Section 5, we complete the proof of Theorem A by verifying the formula for seven squares. Finally, in Section 6, we prove that L_7^* cannot be described with fewer than three packing strategies.

2. THE SQUARE-PACKING FUNCTION

2.1. Basic properties. A packing of a finite or countably infinite nonnegative side vector s in $[0, L]^2$ assigns to every positive entry s_i a square $Q_i = [x_i, x_i + s_i] \times [y_i, y_i + s_i] \subseteq [0, L]^2$, with distinct squares having disjoint interiors. In other words, for all i with $s_i > 0$,

$$(2.1) \quad 0 \leq x_i, y_i \leq L - s_i$$

and, for all distinct i and j with $s_i, s_j > 0$, at least one of the following holds:

$$(2.2) \quad x_i + s_i \leq x_j, \quad x_j + s_j \leq x_i, \quad y_i + s_i \leq y_j, \quad y_j + s_j \leq y_i.$$

Squares in a packing may have touching boundaries. We define $L^*(s)$ to be the infimum of the set of all $L \geq 0$ such that there exists a packing of s in $[0, L]^2$ in the sense above. In order to establish basic properties of L^* , we first recall the Moon–Moser rectangle packing theorem in the case of squares and include a proof for the reader’s convenience.

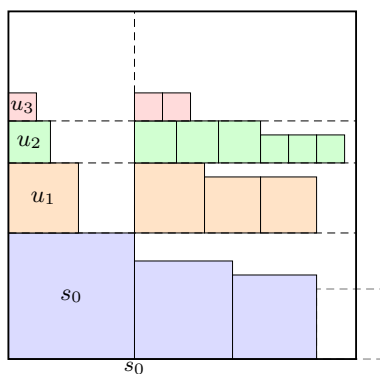


FIGURE 2.1. The shelf packing in the proof of Lemma 2.1 for the sides 9, 7, 6, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2 in the square of side $b = \sqrt{2A} = \sqrt{616}$. The largest square, of side s_0 , and the column squares u_1, u_2, u_3 lie left of the vertical dashed line. The horizontal dashed lines mark the heights at which the shelves begin, and each shelf has the color of its column square. The dashed outline shows that u_1 does not fit at the end of the first shelf.

Lemma 2.1 (Moon–Moser [MM67]). *Every finite or countably infinite family of squares with finite total area A has an axis-parallel packing in a square of side $\sqrt{2A}$.*

Proof. The case $A = 0$ is immediate. Otherwise arrange the squares with positive side lengths in decreasing order $s_0 \geq s_1 \geq \dots$. Let $A = \sum_i s_i^2$ denote the total area and put $b = \sqrt{2A}$, so $b > s_0$. Place the largest square at the lower left corner of $[0, b] \times [0, \infty)$. Reserve the left strip of width s_0 for a column of squares. To its right, place successive squares on *horizontal shelves*: rows of squares placed consecutively from left to right with their bottom edges aligned. Begin the first shelf at height zero. Whenever the next square does not fit in the available width $b - s_0$, put it in the left column immediately above the preceding column square and begin another shelf at the same height (Figure 2.1). Since the sides are decreasing, every square on a shelf is no taller than its column square. Thus, the shelves do not overlap.

Let $u_1, u_2, \dots$ be the sides of the column squares after the first, and w_j the total width occupied by the shelf immediately preceding u_j . Since u_j did not fit, $w_j + u_j > b - s_0$. Every square on that shelf has side at least u_j . Hence the total area of the shelf together with u_j^2 exceeds $u_j(b - s_0)$. These groups of squares are disjoint. Therefore, for each m for which u_m exists,

$$(b - s_0) \sum_{j=1}^m u_j \leq A - s_0^2.$$

Whether the number of shelves is finite or infinite, their total height is consequently at most

$$s_0 + \sum_j u_j \leq s_0 + \frac{A - s_0^2}{b - s_0} \leq b,$$

where the last inequality follows from $(b - s_0)^2 - (A - s_0^2) = (b - 2s_0)^2/2 \geq 0$. Every square is placed at a finite stage, and all shelves lie within the width $b - s_0$ and below height b . Therefore, the whole family fits in $[0, b]^2$. $\square$

Lemma 2.2 (Existence, attainment, and truncation). *A weakly decreasing nonnegative sequence admits a packing in some finite square if and only if it belongs to ℓ_2 . For $s \in \ell_2^\downarrow$, the infimum defining $L^*(s)$ is attained. The function is monotone under coordinatewise increase, positively homogeneous, that is, $L^*(\lambda s) = \lambda L^*(s)$ for all $\lambda \geq 0$, and satisfies*

$$(2.3) \quad \|s\|_2 \leq L^*(s) \leq \sqrt{2} \|s\|_2,$$

$$(2.4) \quad \lim_{n \rightarrow \infty} L_n^*(s) = \sup_n L_n^*(s) = L^*(s).$$

Moreover, for all $n \geq 1$,

$$(2.5) \quad 0 \leq L^*(s) - L_n^*(s) \leq \sqrt{2} \left(\sum_{i=n}^{\infty} s_i^2 \right)^{1/2}.$$

Proof. Disjoint interiors give $\sum_{i < N} s_i^2 \leq L^2$ in any packing. Letting $N \rightarrow \infty$ proves necessity and the lower area bound. Lemma 2.1 gives a packing of the entire sequence with side at most $\sqrt{2}\|s\|_2$, proving sufficiency and the upper area bound.

For a fixed finite family, a bounded minimizing sequence of coordinate vectors has a convergent subsequence. Conditions (2.1) and (2.2) are closed, the latter being a finite union of closed conditions for each pair. This proves finite attainment. Let $B = \sup_n L_n^*(s)$. The preceding bound makes B finite. For each n , choose a packing of the positive entries among the first n in $[0, B]^2$. A diagonal subsequence makes the two coordinates of each fixed square converge. For each fixed pair, containment and nonintersection pass to the limit by the same closedness argument. The limiting coordinates therefore pack the entire sequence in $[0, B]^2$. Conversely, every packing of the sequence packs each truncation. Hence $L^*(s) = B$. This proves optimal attainment and (2.4).

Pack the positive entries among $s_0, \dots, s_{n-1}$ in a square of side $a = L_n^*(s)$ and the tail in a square of side $b \leq \sqrt{2}(\sum_{i \geq n} s_i^2)^{1/2}$. Place the two containing squares side by side. Both fit in $[0, a + b]^2$, and their interiors are disjoint. Thus, $L^*(s) \leq a + b$, which proves (2.5).

Shrinking each square while fixing its lower left corner proves monotonicity. Scaling the coordinates of a packing proves positive homogeneity for positive factors, and $L^*(0) = 0$ handles the zero factor. Deleting or relabelling squares does not increase the required side. $\square$

2.2. Packing strategies and continuity. The combinatorial framework below is closely related to the sequence-pair and constraint-graph method for fixed-orientation rectangles in [MFNK96].

Definition 2.3 (Strategies and complete families). A *strategy* for n squares is a pair $\sigma = (\prec_h, \prec_v)$ of strict partial orders on $\{0, \dots, n-1\}$ such that every pair of distinct indices is comparable in at least one order. The relations prescribe horizontal and vertical separation. A *chain* of σ is a nonempty subset of $\{0, \dots, n-1\}$ totally ordered by either relation. Write $\mathcal{C}(\sigma)$ for the set of all chains of σ and define the *cost* of σ at a vector $v \in [0, \infty)^n$ by $\ell_\sigma(v) = \max_{I \in \mathcal{C}(\sigma)} \sum_{i \in I} v_i$. For $s \in \ell_2^\downarrow$, we write $\ell_\sigma(s)$ for its value at $(s_0, \dots, s_{n-1})$. A family $\mathcal{S}$ of strategies is *complete* if $L_n^*(s) = \min_{\sigma \in \mathcal{S}} \ell_\sigma(s)$ for every $s \in \ell_2^\downarrow$.

For the next lemma, let $t = (t_0, \dots, t_{n-1})$ be a vector of nonnegative side lengths that need not be in decreasing order; we reserve s for nonincreasing sequences. A packing of t has squares $Q_i = [x_i, x_i + t_i] \times [y_i, y_i + t_i]$ for $t_i > 0$.

A packing of t *follows* a strategy σ if, for all i, j with $t_i, t_j > 0$, we have $x_i + t_i \leq x_j$ whenever $i \prec_h j$ and $y_i + t_i \leq y_j$ whenever $i \prec_v j$.

If $t_0, \dots, t_{n-1} > 0$, then a packing of t determines a strategy σ , which we call the *strategy of the packing*: put $i \prec_h j$ when $x_i + t_i \leq x_j$, and $i \prec_v j$ when $y_i + t_i \leq y_j$. Positivity makes each relation irreflexive, the inequalities make it transitive, and (2.2) makes every pair comparable in at least one of them. The packing follows σ .

Lemma 2.4 (Criterion to pack following a strategy). *Let σ be a strategy for n squares and let $t_0, \dots, t_{n-1}$ be nonnegative, not necessarily in decreasing order. There is a packing of t in $[0, L]^2$ following σ if and only if $L \geq \ell_\sigma(t)$.*

Proof. Let I be a chain of $\prec_h$, and list its members with positive sides in increasing order. In a following packing, adding the separation inequalities for consecutive members and the containment inequalities gives $\sum_{i \in I} t_i \leq L$. The vertical argument is identical.

Conversely, for each j let x_j be the largest sum $\sum_{i \in I \setminus \{j\}} t_i$ over chains I of $\prec_h$ with greatest element j , and define y_j using chains of $\prec_v$. Singleton chains give $x_j, y_j \geq 0$, and $x_j + t_j, y_j + t_j \leq \ell_\sigma(t)$. If $i \prec_h j$, extend a maximizing chain ending at i by j to obtain $x_j \geq x_i + t_i$. Likewise $y_j \geq y_i + t_i$ for $i \prec_v j$. Since every pair is comparable in at least one order, these coordinates give the desired packing. $\square$

Proposition 2.5 (Strategy representation). *For every $n \geq 1$ and every $s \in \ell_2^\downarrow$,*

$$(2.6) \quad L_n^*(s) = \min_{\sigma \in \mathfrak{S}_n} \ell_\sigma(s) = \min_{\sigma \in \mathfrak{S}_n} \max_{I \in \mathcal{C}(\sigma)} \sum_{i \in I} s_i,$$

where $\mathfrak{S}_n$ is the finite set of all strategies on n squares. In particular, $L_n^*(s)$ is a continuous, positively homogeneous, and piecewise linear function of $(s_0, \dots, s_{n-1})$, and its value is always a sum of input side lengths, including when some entries are zero.

Proof. Each strategy gives a packing of its own cost by Lemma 2.4, proving the upper inequality. Conversely, if $s_0, \dots, s_{n-1} > 0$, then a packing in $[0, L]^2$ follows its strategy, whose cost is at most L by the same lemma. Since there are only finitely many pairs of relations on a finite set, minimization gives (2.6).

If some entries vanish, the longest-chain construction still packs the positive squares. Conversely, take the separation orders of a packing of the positive squares, then append the zero labels, in increasing order of index, after the positive ones in the horizontal order. This adds nothing to any chain sum, and proves the formula for every $s \in \ell_2^\downarrow$. It also proves continuity and piecewise linearity. Since every finite maximum and minimum selects one of its arguments, every finite packing optimum equals a subset sum. $\square$

Lemma 2.6 (Continuity). *For every $n \geq 1$ and $s, t \in \ell_2^\downarrow$,*

$$(2.7) \quad |L_n^*(s) - L_n^*(t)| \leq \sum_{i < n} |s_i - t_i| \leq \sqrt{n} \|s - t\|_2.$$

The function L^* is continuous on $\ell_2^\downarrow$ in the ℓ_2 norm. If $s - t \in \ell_1$, then

$$(2.8) \quad |L^*(s) - L^*(t)| \leq \|s - t\|_1.$$

Proof. Proposition 2.5 expresses $L_n^*(s)$ as a finite minimum of maxima of subset sums of $s_0, \dots, s_{n-1}$. Each subset sum is 1-Lipschitz in the ℓ_1 norm; finite maxima and minima preserve a common Lipschitz constant. This proves (2.7).

For $u \in \ell_2^\downarrow$ and $n \geq 1$, write $u^{[n]} = (u_0, \dots, u_{n-1}, 0, 0, \dots)$ for the truncation of u , so that $L_n^*(u) = L^*(u^{[n]})$ and, by (2.5), $|L^*(u) - L_n^*(u)| \leq \sqrt{2} \|u - u^{[n]}\|_2$. Fix $s \in \ell_2^\downarrow$. For any t in that cone and any n ,

$$\begin{aligned} |L^*(s) - L^*(t)| &\leq \sqrt{n} \|s - t\|_2 + \sqrt{2} \|s - s^{[n]}\|_2 + \sqrt{2} \|t - t^{[n]}\|_2 \\ &\leq (\sqrt{n} + \sqrt{2}) \|s - t\|_2 + 2\sqrt{2} \|s - s^{[n]}\|_2. \end{aligned}$$

First choose n to make the final term small, and then choose t close to s . This proves continuity. Finally, (2.7) gives $|L_n^*(s) - L_n^*(t)| \leq \|s - t\|_1$ for every n . Letting $n \rightarrow \infty$ and using (2.4) proves (2.8). $\square$

The arguments below concern packings of actual squares, so they assume that the relevant sides are positive. The next lemma shows that lower bounds and formulas proved under this assumption hold in general.

Lemma 2.7 (Reduction to positive sides). *Let $n \geq 1$, and let f be a continuous real-valued function of $(s_0, \dots, s_{n-1})$.*

- (i) *If $L_n^*(s) \geq f(s)$ for every $s \in \ell_2^\downarrow$ with $s_{n-1} > 0$, then $L^*(s) \geq L_n^*(s) \geq f(s)$ for all $s \in \ell_2^\downarrow$.*
- (ii) *If $L_n^*(s) = f(s)$ for every $s \in \ell_2^\downarrow$ with $s_{n-1} > 0$, then $L^*(s) = f(s)$ for all $s \in \ell_2^\downarrow$.*

Proof. Fix $s \in \ell_2^\downarrow$ and $\varepsilon > 0$, and let s^ε be obtained from s by adding ε to each of $s_0, \dots, s_{n-1}$. Then $s^\varepsilon \in \ell_2^\downarrow$, since $s_{n-1} + \varepsilon \geq s_n$, and $s_{n-1}^\varepsilon \geq \varepsilon > 0$. By (2.7), $|L_n^*(s^\varepsilon) - L_n^*(s)| \leq n\varepsilon$, and $f(s^\varepsilon) \rightarrow f(s)$ as $\varepsilon \rightarrow 0$ by the continuity of f . Letting $\varepsilon \rightarrow 0$ in the inequality or equality at s^ε gives it at s . In case (i), $L^*(s) \geq L_n^*(s)$ by the monotonicity in Lemma 2.2. $\square$

3. PACKING UP TO SIX SQUARES

3.1. Elementary chain estimates. Fix a packing of squares Q_i of sides s_i in $[0, L]^2$. For a set of indices I , call $\sum_{i \in I} s_i$ its *side sum*; the same term applies to the corresponding collection $\{Q_i : i \in I\}$. We say that Q_i lies *left of* Q_j when $x_i + s_i \leq x_j$, and *below* Q_j when $y_i + s_i \leq y_j$. Two squares are *x-disjoint* when one lies left of the other, that is, when the interiors of their horizontal projections are disjoint, and *x-overlapping* otherwise. Define y-disjointness and y-overlap using vertical projections. By (2.2), every pair of squares in a packing is x-disjoint or y-disjoint (possibly both). A *horizontal chain* is a collection of pairwise x-disjoint squares, and a *vertical chain* is a collection of pairwise y-disjoint squares. For positive side lengths, the nonempty horizontal and vertical chains of a packing are exactly the chains of the strategy of the packing defined in Section 2.2.

Lemma 3.1 (Chain inequality). *In a packing of side L , every horizontal or vertical chain has side sum at most L . In particular, $L^*(s) \geq s_0 + s_1$ for every $s \in \ell_2^\downarrow$.*

Proof. The projection intervals of a chain have disjoint interiors in $[0, L]$. Hence the sum of their lengths is at most L . Any two positive squares form a chain on at least one axis. If $s_1 = 0$, then $s_0 + s_1 = s_0 \leq L$ by (2.1). $\square$

3.2. Four and five squares. To check explicit coordinates, we use the elementary coefficient test for nonincreasing side lengths. Set $s_n = 0$. Then

$$(3.1) \quad \sum_{i=0}^{n-1} c_i s_i = \sum_{k=0}^{n-1} \left(\sum_{i=0}^k c_i \right) (s_k - s_{k+1}).$$

Thus, a linear form is nonnegative on every decreasing nonnegative tuple if and only if all its coefficient prefix sums are nonnegative. Necessity follows by taking the first $k+1$ sides equal to one and the rest zero. More generally, each coordinate containment or separation inequality below has the form $aL + \sum_{i=0}^{n-1} c_i s_i \geq 0$ with $a \geq 0$. When L is bounded below by the listed chain sums, the inequality is certified by adding a times one of those sums and applying the prefix-sum test.

Proposition 3.2. *For every $s \in \ell_2^\downarrow$,*

$$(3.2) \quad L_4^*(s) = s_0 + s_1$$

and

$$(3.3) \quad L_5^*(s) = \max(s_0 + s_1, s_2 + s_3 + s_4).$$

In particular, $L_1^*(s) = s_0$ and $L_2^*(s) = L_3^*(s) = s_0 + s_1$.

Proof. By Lemma 2.7, it suffices to prove (3.2) when $s_3 > 0$ and (3.3) when $s_4 > 0$. Lemma 3.1 gives $L_4^*(s) \geq s_0 + s_1$. For $L \geq s_0 + s_1$, take lower left corners $(0, 0)$, $(s_0, 0)$, $(0, L - s_2)$, and $(L - s_3, L - s_3)$. Squares Q_2, Q_3 lie above Q_0, Q_1 , respectively, Q_2 lies left of Q_1 and Q_3 , and Q_0 lies left of Q_1 and Q_3 . These relations and containment in $[0, L]^2$ follow from $L \geq s_0 + s_1$ by the coefficient test (3.1). Thus, $L_4^*(s) \leq s_0 + s_1$. Since $L_n^*(s) = L_4^*(s_0, \dots, s_{n-1}, 0, 0, \dots)$ for $n \leq 3$, the formulas for L_1^*, L_2^*, L_3^* are special cases of (3.2).

For five squares and $L \geq \max(s_0 + s_1, s_2 + s_3 + s_4)$, take lower left corners $(0, 0)$, $(L - s_1, 0)$, $(0, L - s_2)$, $(s_2, L - s_3)$, and $(L - s_4, L - s_4)$ as in the second panel of Figure 1.1. Squares Q_2, Q_3, Q_4 form a horizontal chain along the top edge and lie above Q_0 and Q_1 , while Q_0 lies left of Q_1 . These relations and containment in $[0, L]^2$ follow from the two lower bounds for L by the coefficient test. Thus, $L_5^*(s) \leq \max(s_0 + s_1, s_2 + s_3 + s_4)$. For the reverse inequality, it suffices to show that a packing of five squares contains a horizontal or vertical chain of three squares, since its side sum is at least $s_2 + s_3 + s_4$. Suppose no three of the squares are pairwise x-disjoint. Let A be the set of squares with no square to their left, and let B consist of the remaining squares. Two squares in A are not x-disjoint, since otherwise one would lie left of the other. Two squares $Q_i, Q_j \in B$ are not x-disjoint either: if Q_i lay left of Q_j , then a square Q_k left of Q_i would make Q_k, Q_i, Q_j pairwise x-disjoint. Thus, the squares in each of A and B are pairwise x-overlapping, hence pairwise y-disjoint. One of A, B contains at least three of the five squares, and these form a vertical chain. $\square$

3.3. The six-square construction. Our coordinate formulas will satisfy a fixed choice of horizontal or vertical separation for every pair throughout the parameter range. The next lemma records how these uniform separations determine a strategy and its cost.

Lemma 3.3 (Uniform coordinates define a strategy). *Let $g(s) \geq 0$ be a continuous function of $(s_0, \dots, s_{n-1})$. Suppose coordinates $x_i(s, L), y_i(s, L)$ satisfy containment for every $s \in \ell_2^\downarrow$ and $L \geq g(s)$. Define $i \prec_h j$ when $x_i + s_i \leq x_j$ holds for all such s and*

L , and define the vertical relation similarly. These are strict partial orders. If every pair is comparable in at least one order, they define a strategy σ with $\ell_\sigma(s) \leq g(s)$. If g is a maximum of side sums of chains of this strategy, then $\ell_\sigma(s) = g(s)$.

Proof. Each pointwise separation relation is transitive, because sides are nonnegative. Their intersection is therefore transitive, and it is irreflexive, as one sees at any s with $s_0, \dots, s_{n-1} > 0$. The uniform relations are satisfied by the coordinates at $L = g(s)$. Hence Lemma 2.4 gives the upper bound for the cost. Each specified chain gives the reverse inequality. $\square$

We now give a single strategy for six squares that works for every nonincreasing side vector.

Proposition 3.4 (Six-square construction). *For every $s \in \ell_2^\downarrow$,*

$$L_6^*(s) \leq \max(s_0 + s_1, s_2 + s_3 + s_4, s_1 + s_4 + s_5).$$

More precisely, for L at least the right-hand side, the coordinates

$$(3.4) \quad \begin{array}{c|ccc} i & 0 & 1 & 2 \\ \hline (x_i, y_i) & (0, 0) & (L - s_1, 0) & (0, L - s_2) \\ i & 3 & 4 & 5 \\ \hline (x_i, y_i) & (s_2, L - s_3) & (L - s_4, L - s_4) & (L - s_5, s_1) \end{array}$$

pack $Q_0, \dots, Q_5$ in $[0, L]^2$ and define a strategy whose cost is the right-hand side.

Proof. Substitution verifies containment and a fixed separation for every pair. Each of these inequalities follows from $s_0 \geq \dots \geq s_5$ and the three lower bounds for L by the coefficient test (3.1). The collections $\{Q_0, Q_1\}$ and $\{Q_2, Q_3, Q_4\}$ are horizontal chains, while Q_1, Q_5, Q_4 form a vertical chain in that order. By Lemma 3.3, the coordinates define a strategy whose cost is the right-hand side. Proposition 2.5 gives the upper bound for L_6^* . $\square$

Figure 1.1 illustrates the constructions.

3.4. Four large squares. The five-square lower bound needed only the existence of a three-square chain. For six squares we need more information, and we first set up the method used for all remaining lower bounds.

Suppose that we wish to prove a lower bound of the form $L^*(s) \geq \Sigma$. We assume to the contrary that some packing fits in a square of side $L < \Sigma$. We say that a set of indices I , or the corresponding collection of squares $\{Q_i : i \in I\}$, *clears* the threshold Σ when $\sum_{i \in I} s_i \geq \Sigma$. By Lemma 3.1, such a collection cannot be a horizontal or vertical chain. The proofs below use these exclusions to force overlaps and then analyze the resulting geometry to get a contradiction.

For example, suppose Q_a, Q_b are y-disjoint and $\{a, b, e\}$ clears Σ . Then Q_e must y-overlap Q_a or Q_b , since otherwise the three squares would form a vertical chain with side sum exceeding L . Whenever two squares y-overlap, they must be x-disjoint. We call the

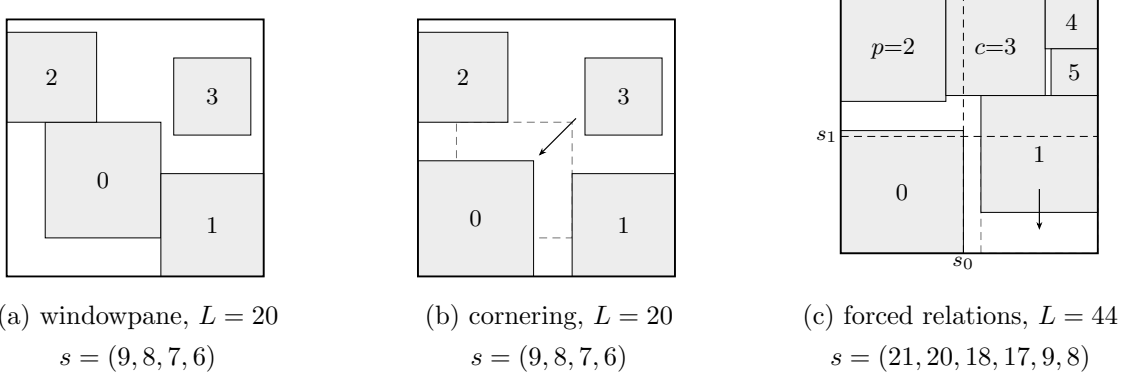


FIGURE 3.1. (a) A windowpane in a square of side $20 < s_1 + s_2 + s_3$: the columns are $\{Q_0, Q_2\}$ and $\{Q_1, Q_3\}$, and the rows are $\{Q_0, Q_1\}$ and $\{Q_2, Q_3\}$. (b) The cornering step of Lemma 3.5: Q_0 has no blocker to its left or below, and moves left and then down from the dashed position to the lower left corner. (c) The conclusions of Lemma 3.6 with $r = N = 5$, where every triple in the hypotheses has side sum at least $s_1 + s_3 + s_5 = 45 > L$. Here $p = 2$ and $c = 3$; the squares Q_4, Q_5 are x -disjoint from Q_p, Q_c , y -overlap both, and lie above height s_1 . Square Q_1 can be lowered to the floor. The dashed lines are $x = s_0$ and $y = s_1$, and the dashed outline is the lowered position of Q_1 .

forced y -overlap with one of Q_a, Q_b a *vertical pin* of Q_e to the pair Q_a, Q_b . A pin to a longer chain, or a horizontal pin, means the analogous necessary overlap.

We will sometimes translate a square to a wall while keeping the other squares fixed. For instance, a *blocker to the right* of Q_i is a square to its right whose vertical projection overlaps that of Q_i : moving Q_i right would eventually meet it. If there is no such blocker, Q_i can be moved to the right wall without disturbing the packing. Blockers in other directions are defined analogously. Reflections and interchanges of the axes move the entire packing. Because clearing depends only on the side lengths, an exclusion based on a clearing set remains valid during any motion that preserves a packing.

If every triple among the four largest squares clears Σ , their projections must have a constrained arrangement, which we call a *windowpane* (Figure 3.1(a)). Its “columns” will be x -overlapping pairs and its “rows” y -overlapping pairs. With an additional clearing hypothesis, this structure also lets us move the largest square into a corner of the full packing (Figure 3.1(b)). Both facts will be used again for seven squares.

Lemma 3.5 (Windowpane and cornering). *Fix a packing of side $L < \Sigma$, and suppose every triple among Q_0, Q_1, Q_2, Q_3 clears Σ . The four squares split into two x -overlapping pairs, called columns, and into two y -overlapping pairs, called rows. No square x -overlaps both members of the other column, or y -overlaps both members of the other row.*

If, in addition, every triple $\{0, b, j\}$ with $b \in \{1, 2, 3\}$ and $j \notin \{0, b\}$ in the full packing clears Σ , then Q_0 can be translated to a corner in that full packing.

Proof. Make a graph with vertices $0, 1, 2, 3$, joining two indices when their squares are x-disjoint. The graph has no triangle, since three pairwise x-disjoint squares would form a clearing horizontal chain. Since it has only four vertices, it has no odd cycle and is bipartite. Each of its two parts consists of pairwise x-overlapping squares. Such squares are pairwise y-disjoint, and a part cannot contain three indices, since a vertical triple would clear Σ . Hence the parts both have size two, giving the two columns. Repeating the argument with the coordinate axes interchanged gives the two rows. A square that x-overlapped both members of the other column would form a forbidden vertical triple with them. The row assertion follows by symmetry.

For the cornering statement, consider blockers on opposite sides of Q_0 in the horizontal direction. They are x-disjoint from Q_0 and from each other. Together with Q_0 they form a horizontal chain. If either blocker is among Q_1, Q_2, Q_3 , this triple clears Σ . If both are smaller squares, there is a square Q_b , $b \in \{1, 2, 3\}$, x-disjoint from Q_0 : otherwise Q_0 and both squares of the other column would form a forbidden vertical triple. Square Q_b is on one side of Q_0 and, together with the blocker on the opposite side, forms a clearing horizontal triple with Q_0 . Thus, there cannot be blockers on both sides. Translate Q_0 to a vertical wall, and repeat the argument vertically to reach a corner; reflect the packing if needed. All the side-sum exclusions remain valid during these motions. $\square$

The next lemma concerns the $r+1$ largest squares $Q_0, \dots, Q_r$ of a packing of $Q_0, \dots, Q_N$, and its hypotheses involve only indices in $\{0, \dots, r\}$. In the proof we remove $Q_{r+1}, \dots, Q_N$, lower Q_1 in the resulting packing of $Q_0, \dots, Q_r$, derive relations among squares whose positions have not changed, and then use those relations in the original packing. The auxiliary translation need not be possible when $Q_{r+1}, \dots, Q_N$ are restored. Figure 3.1(c) illustrates the conclusions.

Lemma 3.6 (Relations forced by four large squares). *Suppose $Q_0 = [0, s_0]^2$ in a packing of $Q_0, \dots, Q_N$ of side $L < \Sigma$, where $N \geq r \geq 4$. Assume that the following triples of indices in $\{0, \dots, r\}$ clear Σ :*

- (i) *every triple among $0, 1, 2, 3$;*
- (ii) *$\{0, 1, j\}$ for $2 \leq j \leq r$;*
- (iii) *$\{0, b, m\}$ and $\{1, b, m\}$ for $b \in \{2, 3\}$ and $4 \leq m \leq r$.*

After possibly interchanging axes, $x_1 \geq s_0$ and $y_1 < s_0$. There are distinct $p, c \in \{2, 3\}$ such that p x-overlaps 0 , c x-overlaps 1 , and c lies above 1 . Moreover, in the original full packing:

- (a) *p, c are x-disjoint, with p left of c ;*
- (b) *each $m \in \{4, \dots, r\}$ is x-disjoint from both p, c and y-overlaps both of them;*
- (c) *$y_m \geq s_1$ for every such m .*

If $r = N$, square Q_1 can be moved to the floor in the full packing.

Proof. Apply the windowpane statement and choose axes so that $x_1 \geq s_0$. Suppose first that $y_1 \geq s_0$. Let p and q be the column and row partners of 0 ; they are $2, 3$ in some

order. Then p lies above 0, q lies right of 0, and 1 lies right of p and above q . These relations follow from the windowpane, since 1, p lie in a row and 1, q in a column, together with $x_p < s_0 \leq x_1$ and $y_q < s_0 \leq y_1$.

Use $m = 4$. Its vertical pin to Q_0, Q_1 makes it overlap one of them. If it overlaps 0, it lies right of 0, and the vertical pin to Q_1, Q_q forces an overlap with 1 or q . It is x-disjoint from that square, and both are right of 0, giving a clearing horizontal triple. Otherwise m is y-disjoint from 0 and y-overlaps 1. It is x-disjoint from 1; the horizontal pin to Q_p, Q_1 makes it x-overlap Q_p . The vertical chain Q_0, Q_p, Q_m now clears Σ , a contradiction. Hence $y_1 < s_0$.

Let c be the column partner of 1 and p the column partner of 0. Suppose c lies below 1. Then $y_c + s_c \leq y_1 < s_0$. Hence c lies right of 0, whereas p lies above 0. Square p cannot lie below 1, and cannot lie above it because Q_c, Q_1, Q_p would be a vertical chain. Thus, p y-overlaps 1 and is x-disjoint from it.

Again use $m = 4$. Its vertical pin to Q_c, Q_1 forces an overlap. If it overlaps c but not 1, it lies below 1 and right of 0, and is x-disjoint from c , giving the clearing horizontal chain Q_0, Q_c, Q_m . Therefore, it overlaps 1 and is x-disjoint from it. Since Q_m is x-disjoint from Q_1 , the horizontal pin to Q_0, Q_1 forces it to x-overlap Q_0 . Thus, Q_m is y-disjoint from $Q_0 = [0, s_0]^2$ and lies above it. If m, p are x-disjoint, the horizontal chain Q_1, Q_p, Q_m clears Σ ; otherwise the vertical chain Q_0, Q_p, Q_m clears Σ . Hence c lies above 1.

Now remove $Q_{r+1}, \dots, Q_N$. In the packing of $Q_0, \dots, Q_r$, square 1 can be lowered to the floor with all the other squares fixed. A downward blocker other than 0 would complete a clearing vertical triple with 1, c , while 0 is x-disjoint from 1. In the auxiliary placement, Q_0 and the lowered Q_1 both lie on the floor. Each of $Q_p, Q_4, \dots, Q_r$ is pinned horizontally to Q_0, Q_1 . It therefore x-overlaps one of them, is y-disjoint from it, and lies above it. Since $s_0 \geq s_1$, each of these squares lies above height s_1 . The vertical pins to Q_1, Q_c make these squares y-overlap Q_c , hence be x-disjoint from it. The pins to Q_1, Q_p likewise make each $m \geq 4$ y-overlap p , hence be x-disjoint from it.

All coordinates occurring in (a)–(c) have remained fixed; only the vertical coordinate of 1 changed. Thus, these assertions hold in the original full packing, regardless of $Q_{r+1}, \dots, Q_N$. Finally, p lies left of c : the reverse order would give $x_c + s_c \leq x_p < s_0 \leq x_1$, contradicting the x-overlap of $c, 1$. If $r = N$, the auxiliary motion is a motion of the full packing. $\square$

3.5. A column bound and the six-square formula.

Proposition 3.7 (Column bound). *For every $n \geq 6$ and $s \in \ell_2^\downarrow$,*

$$(3.5) \quad L^*(s) \geq L_n^*(s) \geq \min(s_1 + s_3 + s_{n-1}, P_n(s)), \quad \text{where } P_n(s) = s_1 + \sum_{i=4}^{n-1} s_i.$$

Proof. By Lemma 2.7, it suffices to prove the inequality for L_n^* when $s_{n-1} > 0$. Set

$$D = s_1 + s_3 + s_{n-1} \quad \text{and} \quad Z = s_2 + s_3 + s_{n-2} + s_{n-1},$$

and suppose that $L < \Sigma := \min(D, P_n, Z)$. Every triple of indices among $0, 1, 2, 3$ clears Σ ; so do $\{0, b, j\}$ with $b \in \{1, 2, 3\}$ and any other index j , and $\{1, b, m\}$ with $b \in \{2, 3\}$ and $m \geq 4$. Each of these triples has side sum at least D , because s is nonincreasing. Lemma 3.5 corners Q_0 , and Lemma 3.6 with $r = N = n - 1$ lowers Q_1 to the floor in the full packing. It also makes Q_2, Q_3 x-disjoint from one another and from every square with index at least 4, all of which lie above Q_1 .

If two of the squares $Q_4, \dots, Q_{n-1}$ are x-disjoint, they and Q_2, Q_3 form a horizontal chain of sum at least Z . Otherwise these squares are pairwise x-overlapping and form a vertical chain. With Q_1 below all of them, this chain has sum P_n . Either chain contradicts $L < \Sigma$. Thus, $L \geq \min(D, P_n, Z)$ in every packing.

The auxiliary term Z is removable. If $\min(D, P_n) \leq s_0 + s_1$, use Lemma 3.1. Otherwise $D > s_0 + s_1$, which gives $s_3 + s_{n-1} > s_0 \geq s_1$ and $s_2 + s_{n-2} \geq s_3 + s_{n-1} > s_1$. It follows that $Z > D$, proving (3.5). $\square$

Theorem 3.8. *For every $s \in \ell_2^\downarrow$,*

$$(3.6) \quad L_6^*(s) = \max(s_0 + s_1, s_2 + s_3 + s_4, s_1 + s_4 + s_5).$$

Proof. Proposition 3.4 proves the upper bound. Lemma 3.1 gives $L_6^* \geq s_0 + s_1$, and $L_6^* \geq L_5^* \geq s_2 + s_3 + s_4$ by (3.3). Proposition 3.7 with $n = 6$ gives the remaining lower bound $L_6^* \geq \min(s_1 + s_3 + s_5, s_1 + s_4 + s_5) = s_1 + s_4 + s_5$, because $s_3 \geq s_4$. This proves (3.6). $\square$

4. SHARP BOUNDS ON THE NUMBER OF SIDES IN A SUM

Proposition 2.5 guarantees that every value $L_n^*(s)$ is some subset sum. We now determine how many sides that sum may need. For $s \in \ell_2^\downarrow$, call $I \subseteq \{0, \dots, n-1\}$ a *representing set* if $L_n^*(s) = \sum_{i \in I} s_i$, and put $q_n(s) = \min\{|I| : I \text{ represents } s\}$. We prove $\max_{s \in \ell_2^\downarrow} q_n(s) = n - 3$ for $n \geq 6$.

4.1. Representing sums with at most $n - 3$ sides. We begin with two placements that work for any number of squares.

Proposition 4.1 (Two general constructions). *For $n \geq 6$ and P_n as in Proposition 3.7, one has*

$$(4.1) \quad L_n^*(s) \leq \max(s_0 + s_1, s_2 + s_3 + s_4, P_n).$$

Also,

$$L_n^*(s) \leq \max\left(s_0 + s_1, s_0 + s_2 + s_3, \sum_{\substack{2 \leq i < n \\ i \text{ even}}} s_i\right).$$

In particular, if $s_2 + s_3 \leq s_1$, then

$$(4.2) \quad L_n^*(s) \leq \max\left(s_0 + s_1, \sum_{\substack{2 \leq i < n \\ i \text{ even}}} s_i\right).$$

Proof. For the first construction, set $c = \max(s_0, s_2 + s_3)$ and place $Q_0, \dots, Q_3$ with respective lower left corners

$$(0, 0), \quad (s_0, 0), \quad (0, s_0), \quad (s_2, s_0).$$

For $4 \leq i \leq n-1$, put $(x_i, y_i) = (c, s_1 + \sum_{j=4}^{i-1} s_j)$. The column squares are mutually y-disjoint, lie above Q_1 , and lie right of Q_0, Q_2, Q_3 . The shelf squares Q_2, Q_3 are x-disjoint and lie above the two bottom squares. The horizontal extent is $\max(s_0 + s_1, c + s_4) = \max(s_0 + s_1, s_2 + s_3 + s_4)$, since $s_4 \leq s_1$. The vertical extent is $\max(s_0 + s_2, P_n) \leq \max(s_0 + s_1, P_n)$.

For the second construction, retain Q_0, Q_1 at the bottom. Place the even-indexed squares $Q_2, Q_4, \dots$ consecutively from left to right on the shelf at height s_0 , and the odd-indexed squares $Q_3, Q_5, \dots$ on the shelf at height $s_0 + s_2$. Their heights are at most s_2 and s_3 , respectively. By termwise comparison, the odd-indexed shelf is no wider than the even-indexed one. The bounds follow from the horizontal and vertical extents. $\square$

We next show that representing sets with at least $n-2$ elements can be avoided. Complementation is convenient: for any index sets I, J , the inequality $\sum_{i \in I} s_i < \sum_{j \in J} s_j$ is equivalent to $\sum_{j \in J^c} s_j < \sum_{i \in I^c} s_i$, since a subset sum and its complementary sum add to the total side sum.

Theorem 4.2 (Representing sums with at most $n-3$ sides). *Let $n \geq 6$ and $s \in \ell_2^\downarrow$ with $s_{n-1} > 0$. If a representing set I has $|I| \geq n-2$, then $L_n^*(s) = s_0 + s_1$. For every $s \in \ell_2^\downarrow$, including vectors with zero entries, $q_n(s) \leq n-3$.*

Proof. Let $J = I^c$. Then $|J| \leq 2$, so $\sum_{j \in J} s_j \leq s_0 + s_1$ and

$$(4.3) \quad \sum_{i \in I} s_i \geq \sum_{i=2}^{n-1} s_i.$$

The chain inequality gives $\sum_{i \in I} s_i = L_n^* \geq s_0 + s_1$. Suppose it is strictly larger; we obtain a packing of side strictly less than $\sum_{i \in I} s_i$.

If $s_1 < s_2 + s_3$, use Proposition 4.1. Its first term is smaller than $\sum_{i \in I} s_i$ by assumption. Its second is smaller by (4.3), since the tail beginning with s_5 is nonempty and positive. Its third is smaller by complementation, since $\sum_{j \in J} s_j \leq s_0 + s_1 < s_0 + s_2 + s_3$. Hence all three terms of (4.1) are smaller than $\sum_{i \in I} s_i$.

If $s_2 + s_3 \leq s_1$, use (4.2). The first term is smaller than $\sum_{i \in I} s_i$. The even-indexed sum beginning with s_2 is strictly smaller than $\sum_{i=2}^{n-1} s_i \leq \sum_{i \in I} s_i$, because the latter also includes $s_3 > 0$. This is again a contradiction.

Thus, when all sides are positive, a long representing set forces $L_n^* = s_0 + s_1$, and it may be replaced by $\{0, 1\}$. Since $2 \leq n-3$ and some representing set exists by Proposition 2.5, this proves $q_n(s) \leq n-3$ for positive vectors. If $s_{n-1} = 0$, let $k < n$ be the number of positive entries among $s_0, \dots, s_{n-1}$. The assertion follows by applying the result to the positive prefix when $k \geq 6$, or by the formulas for at most five squares when $k \leq 5$. In all cases the resulting representing set has at most $n-3$ entries. $\square$

The two constructions serve different parameter ranges. For example, with $n = 10$ and sides $(5, 5, 2, \dots, 2)$, the column construction gives an upper bound of 17, whereas the two-shelf construction gives the exact value 10.

4.2. Sharpness and consequences. We now show that the bound $n - 3$ is attained on an open set of side vectors.

Theorem 4.3 (Sharpness). *For every $n \geq 6$, there is a nonempty open set U of strictly decreasing positive n -tuples such that, for all $s \in \ell_2^\downarrow$ with $(s_0, \dots, s_{n-1}) \in U$,*

$$L_n^*(s) = P_n(s) = s_1 + \sum_{i=4}^{n-1} s_i \quad \text{and} \quad q_n(s) = n - 3.$$

Proof. Let $n \geq 6$, set $m = n - 5$, and consider

$$(4.4) \quad s_0 = s_1 = 8m + 4, \quad s_2 = s_3 = 8m + 1, \quad s_4 = \dots = s_{n-1} = 8.$$

There are $m + 1$ sides equal to 8. At this input, $P_n = 16m + 12$ and

$$s_0 + s_1 = P_n - 4, \quad s_2 + s_3 + s_4 = P_n - 2, \quad s_1 + s_3 + s_{n-1} = P_n + 1.$$

The column construction and the column bound of Proposition 3.7 give $L_n^* = P_n$. Figure 4.1 shows the member of this family with $m = 4$. The three displayed comparisons remain strict in a neighborhood of this vector. Intersecting that neighborhood with the cone of strictly decreasing positive n -tuples gives a nonempty open set on which $L_n^* = P_n$.

For each subset I of cardinality less than $n - 3$, the linear forms P_n and $\sum_{i \in I} s_i$ have different coefficient vectors. Their equality defines a proper hyperplane. Since there are

		8	8
2	3	7	7
		6	6
		5	5
		4	4
0	1		1

$$L_9^* = 76$$

$$s = (36, 36, 33, 33, 8, 8, 8, 8, 8)$$

FIGURE 4.1. The member of the sharpness family (4.4) with $m = 4$. The six-term vertical chain has side sum 76, whereas $s_0 + s_1 = 72$, $s_2 + s_3 + s_4 = 74$, and $s_1 + s_3 + s_8 = 77$. The lower bound proves that the column cost is optimal.

only finitely many such subsets, we may choose a smaller open ball that avoids all of these hyperplanes. On that ball, P_n has no representation using fewer than $n - 3$ sides. $\square$

Finally, we deduce consequences for linear components, for minimum-of-maxima formulas, and for Lipschitz constants. A *linear component* means a linear function of $(s_0, \dots, s_{n-1})$ that agrees with $L_n^*(s)$ on a nonempty open set of strictly decreasing positive n -tuples.

Corollary 4.4 (Minimum-of-maxima formulas with at most $n - 3$ sides per sum). *For $n \geq 6$, every linear component is a subset sum with at most $n - 3$ entries. There is a finite minimum-of-maxima formula for $L_n^*(s)$ in $s_0, \dots, s_{n-1}$ whose constituent sums all have that cardinality bound. No such representation using only shorter subset sums exists.*

Proof. The finite strategy formula subdivides the cone of nonincreasing nonnegative n -tuples into finitely many polyhedral regions on which $L_n^*(s)$ is a subset sum. A linear component agrees with one of these subset sums on an open subset of its region, and hence is a subset sum. On a full-dimensional region, Theorem 4.2 excludes a component of cardinality at least $n - 2$: it would equal $s_0 + s_1$ on an open set, hence as a linear form, contrary to its different coefficient vector. Boundary values follow by continuity from the full-dimensional regions.

The lattice representation theorem for continuous piecewise-linear functions on closed convex sets [Ovc02] expresses such a function as a finite maximum of minima of its linear components. Apply it to $-L_n^*$ on the closed convex cone of nonincreasing nonnegative n -tuples. Since the linear components of $-L_n^*$ are the negatives of those of L_n^* , this expresses L_n^* as a finite minimum of maxima of its linear components. The components are the short sums just identified. Finally, any finite minimum-of-maxima expression selects one of its constituent values at every input. Theorem 4.3 therefore rules out using only shorter subset sums. $\square$

Corollary 4.5 (Sharp Euclidean Lipschitz constants). *For every $n \geq 6$,*

$$|L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3} \|s - t\|_2 \quad \text{for all } s, t \in \ell_2^\downarrow,$$

and the constant $\sqrt{n-3}$ cannot be decreased. In particular L^* is not globally Lipschitz on $\ell_2^\downarrow$ in the ℓ_2 metric.

Proof. A sum of at most $n - 3$ coordinates has gradient of Euclidean norm at most $\sqrt{n-3}$. Since finite minima and maxima preserve a common Lipschitz constant, Corollary 4.4 gives

$$|L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3} \left(\sum_{i < n} (s_i - t_i)^2 \right)^{1/2} \leq \sqrt{n-3} \|s - t\|_2.$$

For sharpness, let U be the open set in Theorem 4.3, on which $L_n^* = P_n$, and let $e \in \mathbb{R}^n$ have entries $e_1 = 1$, $e_i = 1$ for $4 \leq i \leq n - 1$, and $e_i = 0$ otherwise. Choose an n -tuple $s \in U$, identified with its zero-padded sequence. Since U is open, there is $h > 0$ such that $t = s + he$ also belongs to U . Then

$$L_n^*(t) - L_n^*(s) = P_n(t) - P_n(s) = (n-3)h \quad \text{and} \quad \|t - s\|_2 = \sqrt{n-3}h.$$

The ratio of the change in value to the distance is therefore $\sqrt{n-3}$. Since s and t have zero tails, $L^*(s) = L_n^*(s)$ and $L^*(t) = L_n^*(t)$. Thus, a Lipschitz constant for L^* on $\ell_2^\downarrow$ would have to be at least $\sqrt{n-3}$ for every n . $\square$

Together, Theorem 4.2, Theorem 4.3, and the two corollaries prove Theorem B.

5. SEVEN SQUARES

5.1. Three strategies. Put

$$(5.1) \quad \begin{aligned} W &= s_3 + s_4 + s_5 + s_6, & H &= s_1 + s_2 + s_5, \\ C_0 &= s_0 + s_5 + s_6, & C_3 &= s_1 + s_3 + s_6, & R &= s_1 + s_4 + s_5 + s_6. \end{aligned}$$

and

$$(5.2) \quad F_7 = \max(L_6^*, \min(\max(W, H), \max(C_0, C_3), R)).$$

The three strategies are defined by the coordinates in Table 5.1. They are named for the position of Q_6 : against the top edge, in the interior, and against the right edge.

Proposition 5.1 (Three seven-square strategies). *The coordinates in Table 5.1 define strategies $\sigma_{\text{top}}, \sigma_{\text{center}}, \sigma_{\text{right}}$ with costs*

$$\ell_{\text{top}} = \max(L_6^*, W, H), \quad \ell_{\text{center}} = \max(L_6^*, C_0, C_3), \quad \ell_{\text{right}} = \max(L_6^*, R).$$

For each strategy and every L at least its cost, the corresponding row of Table 5.1 gives a packing in $[0, L]^2$. Consequently, for every $s \in \ell_2^\downarrow$,

$$L_7^*(s) \leq \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}) = F_7(s).$$

Proof. Containment and pairwise separation follow from the coefficient test (3.1), and Lemma 3.3 gives the strategies. For example, in the center strategy Q_6 lies right of Q_0 because $x_6 = L - s_5 - s_6 \geq s_0$ when $L \geq C_0$, and below Q_3 because $y_6 + s_6 = s_1 + s_6 \leq L - s_3 = y_3$ when $L \geq C_3$. This is how C_0 and C_3 enter the cost. The three strategy costs are the maxima of the following chain sums:

	horizontal	vertical
top	$s_0 + s_1, s_2 + s_3 + s_4, s_3 + s_4 + s_5 + s_6$	$s_1 + s_2 + s_5$
center	$s_0 + s_1, s_2 + s_3 + s_4, s_0 + s_5 + s_6$	$s_1 + s_4 + s_5, s_1 + s_3 + s_6$
right	$s_0 + s_1, s_2 + s_3 + s_4$	$s_1 + s_4 + s_5 + s_6.$

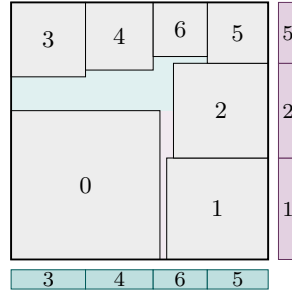
Each term of (3.6) that is not displayed is dominated by a displayed sum, which gives the three costs. The identity $\max(a, \min_j b_j) = \min_j \max(a, b_j)$ shows that their minimum is F_7 , and Proposition 2.5 gives the upper bound for L_7^* . $\square$

Figure 1.2 shows chains attaining the costs.

strategy	(x_0, y_0)	(x_1, y_1)	(x_2, y_2)	(x_3, y_3)
	(x_4, y_4)	(x_5, y_5)	(x_6, y_6)	
top	$(0, 0)$	$(L - s_1, 0)$	$(L - s_2, s_1)$	$(0, L - s_3)$
	$(s_3, L - s_4)$	$(L - s_5, L - s_5)$	$(s_3 + s_4, L - s_6)$	
center	$(0, 0)$	$(L - s_1, 0)$	$(0, L - s_2)$	$(s_2, L - s_3)$
	$(L - s_4, L - s_4)$	$(L - s_5, s_1)$	$(L - s_5 - s_6, s_1)$	
right	$(0, 0)$	$(L - s_1, 0)$	$(0, L - s_2)$	$(s_2, L - s_3)$
	$(L - s_4, L - s_4)$	$(L - s_5, s_1)$	$(L - s_6, s_1 + s_5)$	

TABLE 5.1. Coordinates of the lower left corners of $Q_0, \dots, Q_6$ for the three strategies top, center, and right, as functions of the tuple s and the side L .

5.2. Cornering and a perpendicular chain cover. For seven squares, a more general cornering argument replaces the four-square windowpane. When every triple containing the largest square clears Σ , we can move that square to a corner. Each remaining square must then lie to its right or above it. Those two groups form chains in perpendicular directions (Figure 5.1), and the following lemmas show how to select a pair of chains that accounts for all the remaining squares. This will force a four-square chain in the proof of Proposition 5.4.



corner frame, $L = 38$
 $s = (22, 15, 14, 11, 10, 9, 8)$

FIGURE 5.1. The corner frame of Lemma 5.2 for the top strategy, where every triple through Q_0 has side sum at least $s_0 + s_5 + s_6 = 39 > L$. The right arm $\{1, 2, 5\}$ lies in the purple region $x \geq s_0$, the upper arm $\{3, 4, 5, 6\}$ lies in the teal region $y \geq s_0$, and the arms share Q_5 .

Lemma 5.2 (Corner frame). *Fix a packing of $Q_0, \dots, Q_N$ of side $L < \Sigma$ and an index $m \leq N$, and suppose every triple of indices in $\{0, \dots, m\}$ that contains 0 clears Σ . Then Q_0 can be translated to a corner in the packing of $Q_0, \dots, Q_m$ obtained by removing $Q_{m+1}, \dots, Q_N$. After reflection, let $Q_0 = [0, s_0]^2$ and put*

$$\mathcal{R} = \{i \in \{1, \dots, m\} : x_i \geq s_0\}, \quad \mathcal{U} = \{i \in \{1, \dots, m\} : y_i \geq s_0\}.$$

The arms cover $\{1, \dots, m\}$. The right arm is a vertical chain with pairwise x -overlaps, the upper arm is a horizontal chain with pairwise y -overlaps, and $|\mathcal{R} \cap \mathcal{U}| \leq 1$. If Q_0 is already cornered in the packing of $Q_0, \dots, Q_N$, these conclusions about $Q_1, \dots, Q_m$ require no motion.

Proof. Blockers on both sides of the horizontal band through Q_0 would form a clearing horizontal triple with it. One side is therefore free; move Q_0 to a vertical wall. Repeat vertically and reflect. Every other square then lies right of Q_0 or above it. Two x -disjoint members of $\mathcal{R}$ would form a clearing horizontal triple with Q_0 . They therefore x -overlap and are y -disjoint. The upper-arm assertion is symmetric. Two members in the intersection would be required both to x -overlap, as right-arm members, and to be x -disjoint, as upper-arm members. $\square$

Lemma 5.3 (Perpendicular covering chains). *In the frame of Lemma 5.2, with $m \geq 1$, there are a vertical chain $\mathcal{V}$ and a horizontal chain $\mathcal{H}$ such that $\mathcal{V} \cup \mathcal{H} = \{1, \dots, m\}$ and $\mathcal{V} \cap \mathcal{H}$ consists of one index. In particular, for $m = 6$ some four squares form a chain.*

Proof. If one arm is empty, the other arm is a chain containing every index, and we pair it with a singleton. If the arms meet, they are the required chains. Otherwise the arms are disjoint and nonempty. Let r be topmost in $\mathcal{R}$ and u rightmost in $\mathcal{U}$. Since

$$x_r \geq s_0 > x_u, \quad y_u \geq s_0 > y_r,$$

either r lies right of u , or u lies above r . In the first case, since u is the rightmost member of $\mathcal{U}$ and lies left of r , every member of $\mathcal{U}$ lies left of r . Thus, $\mathcal{R}$ and $\mathcal{U} \cup \{r\}$ are the required chains. In the second case, $\mathcal{R} \cup \{u\}$ and $\mathcal{U}$ are the required chains by symmetry. The two chain cardinalities sum to $m + 1$. Hence for $m = 6$ one of them is at least four. $\square$

5.3. Proof of the seven-square formula. Put $W_{24} = s_2 + s_4 + s_5 + s_6$.

Proposition 5.4 (Three lower bounds). *For every $s \in \ell_2^\downarrow$,*

$$(5.3) \quad L_7^*(s) \geq \min(C_0, W),$$

$$(5.4) \quad L_7^*(s) \geq \min(C_3, R),$$

$$(5.5) \quad L_7^*(s) \geq \min(C_0, H, W_{24}).$$

Proof. By Lemma 2.7, we may assume $s_6 > 0$. The first inequality illustrates the method: when every triple through Q_0 clears the threshold, we can corner Q_0 , and the resulting arms then force a long chain. For (5.3), put $\Sigma = \min(C_0, W)$ and suppose a packing has side $L < \Sigma$. Every triple through Q_0 has side sum at least C_0 and clears Σ . Corner Q_0 and apply Lemma 5.3 to the six remaining squares. The resulting four-square chain has side sum at least W , a contradiction. Inequality (5.4) is Proposition 3.7 with $n = 7$.

For (5.5), suppose $L < \Sigma := \min(C_0, H, W_{24})$. Corner Q_0 and form the corner frame on $Q_1, \dots, Q_6$. Every four-subset of these indices except $\{3, 4, 5, 6\}$ has side sum at least W_{24} and clears Σ . Thus, the four-chain supplied by Lemma 5.3 must be exactly $\{3, 4, 5, 6\}$;

interchange axes to make it horizontal. At most one of its members lies in the right arm. Hence at least three lie in the upper arm. Neither Q_1 nor Q_2 can lie in the upper arm, because it and these three would form a clearing horizontal quadruple. Thus, Q_1, Q_2 lie in the exclusive right arm and form a vertical chain, whose upper member Q_t has $y_t < s_0$.

For $k = 3, 4, 5$, the triple $\{Q_1, Q_2, Q_k\}$ has side sum at least H and clears Σ . Hence Q_k cannot lie in the right arm. If Q_k x-overlapped Q_t , it would lie above Q_t , since $y_k \geq s_0 > y_t$, and the squares Q_1, Q_2, Q_k would form a clearing vertical chain. Hence Q_3, Q_4, Q_5 are all x-disjoint from Q_t . They are mutually x-disjoint as members of the forced four-chain. The quadruple $\{t, 3, 4, 5\}$ differs from $\{3, 4, 5, 6\}$ and therefore clears Σ , a contradiction. $\square$

Theorem 5.5. *For every $s \in \ell_2^\downarrow$,*

$$L_7^*(s) = F_7(s) = \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}),$$

where $\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}$ are the costs in Proposition 5.1.

Proof. Proposition 5.1 gives $L_7^* \leq F_7 = \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}})$. For the reverse inequality, note that $L_7^* \geq L_6^*$ by Lemma 2.2. If $F_7 = L_6^*$, this suffices. Otherwise $F_7 > L_6^* \geq s_0 + s_1$, and $F_7 \leq R$, $F_7 \leq \max(C_0, C_3)$, and $F_7 \leq \max(W, H)$. If $C_3 \geq F_7$, use (5.4). Otherwise $C_0 \geq F_7$. If $W \geq F_7$, use (5.3). Otherwise $H \geq F_7$. Since $C_0 \geq F_7 > s_0 + s_1$, we have $s_5 + s_6 > s_1$, whence

$$W_{24} - H = s_4 + s_6 - s_1 \geq s_5 + s_6 - s_1 > 0.$$

All three terms in (5.5) are then at least F_7 . Thus, $L_7^* \geq F_7$. $\square$

6. MINIMALITY OF CONVEX REPRESENTATIONS

To end the paper, we prove that the complete strategy families of the preceding sections are minimal. Every strategy cost ℓ_σ is a maximum of finitely many linear functions, namely its chain sums, and is therefore convex.

Lemma 6.1 (Counting convex costs). *Let D be a convex cone and $F : D \rightarrow \mathbb{R}$ positively homogeneous. Suppose $p_1, \dots, p_N \in D$ satisfy*

$$(6.1) \quad F(p_i + p_j) > F(p_i) + F(p_j) \quad \text{for all } i < j.$$

Every representation $F = \min_{1 \leq k \leq m} h_k$ with convex h_k on D requires $m \geq N$.

Proof. Each $h_k \geq F$. At each p_i , some member of the finite minimum equals $F(p_i)$. If the same member were exact at p_i, p_j , convexity of h_k and homogeneity of F would give

$$F\left(\frac{p_i + p_j}{2}\right) \leq h_k\left(\frac{p_i + p_j}{2}\right) \leq \frac{F(p_i) + F(p_j)}{2} < \frac{F(p_i + p_j)}{2},$$

a contradiction. The exact members at the N points must be distinct. $\square$

Theorem 6.2. *For $n = 6, 7$, the least numbers m of real-valued convex functions $h_1, \dots, h_m$ on $\ell_2^\downarrow$ with $L_n^*(s) = \min_k h_k(s)$ for every $s \in \ell_2^\downarrow$ are 1 and 3, respectively. These are also the minimum cardinalities of complete strategy families.*

Proof. By Theorems 3.8 and 5.5, the strategy of Proposition 3.4 and the three strategies of Proposition 5.1 form complete families of the stated sizes, and their costs are convex. A nonempty minimum needs at least one member, which proves the six-square case.

For seven squares take

$$p_1 = (39, 38, 32, 31, 14, 13, 12), \quad p_2 = (12, 9, 8, 7, 6, 5, 4), \quad p_3 = (16, 10, 9, 8, 7, 6, 5).$$

Evaluation of (5.2) gives

s	p_1	p_2	p_3
$L_7^*(s)$	77	21	26

and

$$L_7^*(p_1 + p_2) = 101 > 98, \quad L_7^*(p_1 + p_3) = 104 > 103, \quad L_7^*(p_2 + p_3) = 48 > 47.$$

Apply Lemma 6.1. □

Corollary 6.3 (Unit-container parameter sets). *The set $X_6 = \{s \in \ell_2^\downarrow : L_6^*(s) \leq 1\}$ is convex, whereas $X_7 = \{s \in \ell_2^\downarrow : L_7^*(s) \leq 1\}$ is a union of three convex subsets and is not a union of two convex subsets.*

Proof. The set X_6 is a sublevel set of the convex function L_6^* . By Theorem 5.5, X_7 is the union of the sublevel sets of the three convex strategy costs $\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}$. For the last assertion, let p_1, p_2, p_3 be the vectors in the proof of Theorem 6.2, and put $\hat{p}_i = p_i / L_7^*(p_i) \in X_7$. For $i \neq j$, the point $(p_i + p_j) / (L_7^*(p_i) + L_7^*(p_j))$ lies on the segment joining $\hat{p}_i$ and $\hat{p}_j$, and its optimal value $L_7^*(p_i + p_j) / (L_7^*(p_i) + L_7^*(p_j))$ is greater than one. Thus, no convex subset of X_7 contains two of $\hat{p}_1, \hat{p}_2, \hat{p}_3$, whereas one of two sets covering X_7 would contain two of them. □

In the notation of [CKK⁺17], the points $\hat{p}_1, \hat{p}_2, \hat{p}_3$ form a clique in the invisibility graph of X_7 , and the convexity number of X_7 is three.

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This project began in June 2026, when I started actively looking for a formula for L_6^* , and I obtained a complete proof of the formula for L_7^* in July 2026. Throughout this period I used Claude Opus 4.8 (Anthropic) as a research assistant. The candidate formulas came from geometric intuition: I proposed arrangements of the squares, such as shelves, columns, and the configurations that became the three strategies in Figure 1.2, and read candidate terms off the chains that determine the container side. The AI system implemented exact solvers and exhaustive searches over integer side vectors that tested these candidates and returned counterexamples, whose optimal packings suggested further arrangements. I designed the lower-bound method: assume that a packing has side below the asserted bound, use side sums exceeding that side to exclude chains, and follow the forced overlaps to a contradiction. With the assistance of Claude Opus 4.8, this approach was refined into its present form, built from clearing sets, pins, cornering arguments, and a small number of lower bounds combined by arithmetic. In preparing this manuscript,

I used Claude Opus 5.5 (Anthropic) and GPT-6 Astra Pro to review the mathematical arguments and to improve the exposition; Claude Opus 5.5 also assisted with typesetting, figures, and exact-arithmetic checks of the constructions. I have checked all proofs and take full responsibility for the content of this paper.

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