AXIS-PARALLEL SQUARE PACKING WITH AN EXACT FORMULA FOR OPTIMAL PACKING OF SEVEN SQUARES

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Abstract. How large a square is needed to hold axis-parallel squares of prescribed side lengths? We give exact formulas for the optimal side through seven squares, with explicit constructions and geometric lower-bound proofs. For every $n \geq 6$, the optimum for n squares is a sum of at most $n-3$ of the prescribed side lengths. This bound is sharp on an open set of side lengths and yields the optimal Euclidean Lipschitz constant $\sqrt{n-3}$ of the optimum as a function of the n side lengths.

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1. Introduction

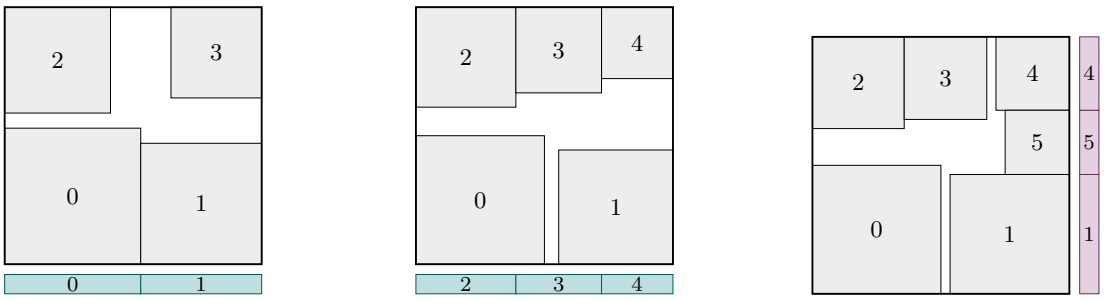
For every sequence $s = (s_i)_{i=0}^{\infty}$ of real numbers with $s_0 \geq s_1 \geq \cdots \geq 0$, let $L^*(s)$ denote the smallest side length of an axis-parallel square that contains axis-parallel squares of side lengths $s_0, s_1, \ldots$ with pairwise disjoint interiors. Rotations are not allowed and sides of length zero signify absent squares. By monotonicity and additivity of area and by a theorem of Moon and Moser [MM67], $L^*(s)$ is finite exactly when $s \in \ell_2$, and $\|s\|_2 \leq L^*(s) \leq \sqrt{2}\,\|s\|_2$. Hence L^* is defined on the cone $\ell_2^{\downarrow}$ of nonincreasing nonnegative sequences in ℓ_2. We identify a finite nonincreasing nonnegative tuple with the sequence in $\ell_2^{\downarrow}$ obtained by appending zeros. For all $n \geq 1$ and $s \in \ell_2^{\downarrow}$, we write

$$L_n^*(s) = L^*(s_0, \ldots, s_{n-1}, 0, 0, \ldots)$$

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$L_4^* = 17$ $L_5^* = 18$ $L_6^* = 28$

$s = (9, 8, 7, 6)$ $s = (9, 8, 7, 6, 5)$ $s = (14, 13, 10, 9, 8, 7)$

FIGURE 1.1. Optimal packings of four, five, and six squares. In the second and third panels, $L_5^* = L_4^* + 1$ and $L_6^* = L_5^* + 1$. A teal bar records the horizontal projections of a chain whose side sum is the side length of the square container; a purple bar on the right similarly records a vertical chain.

for the optimum for the n largest squares. Section 2 shows that L^* is continuous and L_n^* is piecewise linear for each n.

1.1. **Exact formulas.** The exact formulas through seven squares are as follows.

Theorem A. *For every $s \in \ell_2^\downarrow$, we have $L_1^*(s) = s_0$ and $L_n^*(s) = s_0 + s_1$ for $2 \le n \le 4$. For five and six squares,*

$$L_5^*(s) = \max(s_0 + s_1,\ s_2 + s_3 + s_4), \tag{1.1}$$

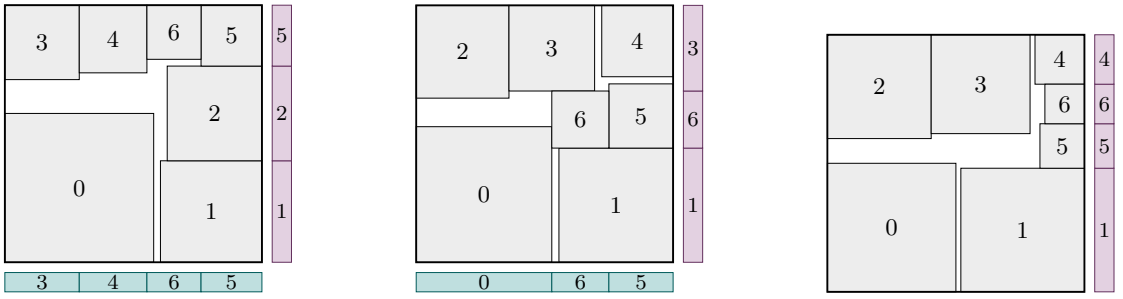
$$L_6^*(s) = \max(s_0 + s_1,\ s_2 + s_3 + s_4,\ s_1 + s_4 + s_5). \tag{1.2}$$

For seven squares,

$$\begin{aligned} L_7^*(s) = \max\big(L_6^*(s), \min\{\, &\max(s_3 + s_4 + s_5 + s_6,\ s_1 + s_2 + s_5), \\ &\max(s_0 + s_5 + s_6,\ s_1 + s_3 + s_6),\ s_1 + s_4 + s_5 + s_6\}\big). \end{aligned} \tag{1.3}$$

A horizontal chain is a family of squares whose horizontal projections have disjoint interiors; vertical chains are defined analogously. The side sum of a collection of squares is the sum of their side lengths. In any packing of axis-parallel squares, the side sum of a chain is at most the container side. The formulas for at most four squares are elementary. The five-square formula (1.1) is implicit in work of Lan, Dósa, Han, Zhou, and Benkő [LDH+13] on packing as many squares as possible into a unit square. In proving that a greedy shelf algorithm is optimal whenever at most five squares fit [LDH+13, Theorem 1], they show that five squares fit into a unit square if and only if the two largest sides sum to at most one and the three smallest sides sum to at most one; the second panel of Figure 1.1 illustrates this. We include a short proof of (1.1), since its chain argument is the starting point for six and seven squares.

The growth of complexity in the formula for L_n^* from $n = 6$ to $n = 7$ is reflected in the minimum size of a strategy family attaining the optimum for every nonincreasing side



top, $L_7^* = 38$
$s = (22, 15, 14, 11, 10, 9, 8)$

center, $L_7^* = 36$
$s = (19, 16, 13, 12, 10, 9, 8)$

right, $L_7^* = 52$
$s = (26, 25, 21, 20, 10, 9, 8)$

FIGURE 1.2. The "top", "center", and "right" packing strategies for seven squares at side vectors where the indicated strategy cost is uniquely optimal among the three strategies. The projection bars record the chains that determine the container side. Each example has $L_7^*(s) = L_6^*(s) + 1$. Labels and colors are as in Figure 1.1.

vector, which rises from one for six squares to three for seven squares; see Figure 1.2. In fact, L_7^* cannot be expressed as the minimum of two convex functions. In a companion paper [Bad26], we prove that L_8^* is the maximum of L_7^* and a minimum of twenty-three maxima of side sums. The proof there uses the methods developed here. At this time, a formula for L_9^* is unknown.

1.2. **How many sides are needed to express the optimum?** Section 2 shows that $L_n^*(s)$ is a piecewise linear function of $(s_0, \dots, s_{n-1})$ whose values are given by a sum of some of the side lengths. The next theorem identifies the largest possible minimum number of sides needed to express the optimum as a subset sum.

Theorem B. *For every $n \geq 6$ and every $s \in \ell_2^\downarrow$, there is a set $I \subseteq \{0, \dots, n-1\}$ with $|I| \leq n-3$ and $L_n^*(s) = \sum_{i \in I} s_i$. This bound is sharp: there is a nonempty open set $U \subseteq \{(s_0, \dots, s_{n-1}) \in \mathbb{R}^n : s_0 > s_1 > \cdots > s_{n-1} > 0\}$ such that*

$$L_n^*(s) = s_1 + \sum_{i=4}^{n-1} s_i \quad \text{for all } s \in \ell_2^\downarrow \text{ with } (s_0, \dots, s_{n-1}) \in U, \tag{1.4}$$

and no subset sum with fewer than $n-3$ entries equals this value. Moreover, $L_n^*(s)$ is given by a finite minimum-of-maxima formula in $s_0, \dots, s_{n-1}$ involving only subset sums of cardinality at most $n-3$, and

$$|L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3}\, \|s - t\|_2 \quad \text{for all } s, t \in \ell_2^\downarrow, \tag{1.5}$$

where the constant $\sqrt{n-3}$ cannot be decreased.

1.3. **Motivation and related packing problems.** The author's interest in L^* arose from joint work with Raanan Schul on Lipschitz parameterizations and rectifiable doubling measures [BS25]. In these constructions a tree organizes successive pieces of a target set,

and squares assigned to the children of a vertex must fit into a square assigned to their parent. The supremum metric is natural on the parameter plane because its balls are axis-parallel squares. Thus, $L_n^*(s)$ is the exact local cost of the packing operation. Replacing an upper estimate by this optimal cost avoids a loss at a step that is repeated throughout a multiscale construction. One useful estimate from [BS25] is

$$L_n^*(s) \le \sum_{j^2<n} s_{j^2} = s_0 + s_1 + s_4 + s_9 + \cdots .$$

It is obtained by successive L-shaped layers and gives $\lceil\sqrt{n}\rceil s$ for n equal sides of length s. The present paper studies the optimal function independently of those applications.

Classical packing theorems often give a guarantee in terms of total area, or optimize the area of a bounding rectangle. The theorem of Moon and Moser [MM67], and its extensions by Meir and Moser [MM68], are fundamental examples; see also Hougardy [Hou11] for packing squares of prescribed total area into a rectangle. For a fixed finite side vector, minimizing the side of a square container is of course equivalent to maximizing the occupied fraction $\sum_i s_i^2/L^2$. Our aim is not a universal or asymptotic density estimate, but the exact dependence of the optimum on every prescribed side. Lower bounds for packing squares and rectangles into squares are analyzed by Caprara, Lodi, Martello, and Monaci [CLMM06].

Another well-known problem asks for the smallest square containing n *unit* squares with arbitrary rotations; Friedman [Fri09] surveys that problem and settles several of its small cases. With fixed orientations, equal sides are elementary: $L_n^*(1,\dots,1) = \lceil\sqrt{n}\rceil$. Indeed, a grid gives the upper bound, and in a packing in $[0,L]^2$ the integer parts of the lower left corners are distinct points of $\{0,\dots,\lfloor L\rfloor-1\}^2$. With free orientations, five unit squares already fit into a square of side $2+1/\sqrt{2}<3$, and that value is optimal [Fri09].

For rectangles of fixed orientations, Murata, Fujiyoshi, Nakatake, and Kajitani [MFNK96] give a finite sequence-pair encoding and construct placements by longest paths in two constraint graphs. Their result applies to objectives nondecreasing in width and height, hence also to their maximum. There are $(n!)^2$ sequence-pairs. Silvanus and Vygen [SV20] represent a placement of n rectangles by assigning to each pair one of the relations west, south, east, or north, and call a set of such representations complete if it represents every placement without overlaps. They proved that the least size of a complete set lies between

$$\Omega\left(\frac{n!}{n^4}\left(4+2\sqrt{2}\right)^n\right) \quad \text{and} \quad O\left(\frac{n!}{n^6}\left(\frac{11+5\sqrt{5}}{2}\right)^n\right).$$

In this paper, a complete family of strategies (Definition 2.3) need only attain $L_n^*(s)$ for every s, rather than represent every placement. The upper bound of Silvanus and Vygen applies to such families, but their lower bounds do not. Indeed, the least sizes of complete families for six and seven squares are 1 and 3 (Theorem 6.2), and for eight squares the least size is twenty-three [Bad26]. Subsection 2.2 gives the partial-order formulation suited to our arguments.

When n is part of the input, deciding whether prescribed squares fit orthogonally into a prescribed square is strongly NP-complete [LTW+90]. For each fixed n, Proposition 2.5 gives a finite formula for L_n^*; the questions are its size and structure.

The rest of the paper is organized as follows. Section 2 develops general properties of L^* and of finite packing strategies. Section 3 introduces essential terminology used in proofs throughout the paper and proves Theorem A for up to six squares. Section 4 proves Theorem B. In Section 5, we complete the proof of Theorem A by verifying the formula for seven squares. Finally, in Section 6, we prove that L_7^* cannot be described with fewer than three packing strategies.

2. The square-packing function

2.1. **Basic properties.** A packing of a finite or countably infinite nonnegative side vector s in $[0, L]^2$ assigns to every positive entry s_i a square $Q_i = [x_i, x_i+s_i]\times[y_i, y_i+s_i] \subseteq [0, L]^2$, with distinct squares having disjoint interiors. In other words, for all i with $s_i > 0$,

$$0 \le x_i, y_i \le L - s_i \tag{2.1}$$

and, for all distinct i and j with $s_i, s_j > 0$, at least one of the following holds:

$$x_i + s_i \le x_j, \quad x_j + s_j \le x_i, \quad y_i + s_i \le y_j, \quad y_j + s_j \le y_i. \tag{2.2}$$

Squares in a packing may have touching boundaries. We define $L^*(s)$ to be the infimum of the set of all $L \ge 0$ such that there exists a packing of s in $[0, L]^2$ in the sense above. In order to establish basic properties of L^*, we first state the shelf packing of Moon and Moser for squares in the form used below and include a proof for the reader's convenience.

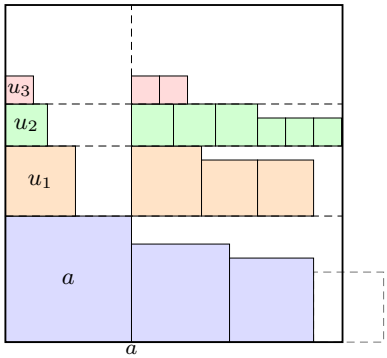


FIGURE 2.1. The shelf packing in the proof of Lemma 2.1 for a square of side $a = 9$ and the sides $7, 6, 5, 5, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2$, of total area $T = 227$, in the square of side $b = 9 + \sqrt{227}$. The square of side a and the column squares u_1, u_2, u_3 lie left of the vertical dashed line. The horizontal dashed lines mark the heights at which the shelves begin, and each shelf has the color of its column square. The dashed outline shows that u_1 does not fit at the end of the first shelf.

Lemma 2.1 (Moon–Moser [MM67, p. 108])**.** *Let $a \geq 0$, and let $\mathcal{F}$ be a finite or countably infinite family of squares, each of side at most a, with finite total area T. Then a square of side a and the squares of $\mathcal{F}$ have an axis-parallel packing in a square of side $a + \sqrt{T}$. In particular, every finite or countably infinite family of squares with total area A has an axis-parallel packing in a square of side $\sqrt{2A}$.*

Proof. The case $T = 0$ is immediate. Otherwise arrange the squares of $\mathcal{F}$ with positive side lengths in decreasing order $t_1 \geq t_2 \geq \cdots$ and put $b = a + \sqrt{T}$. Each t_i is at most a and at most $\sqrt{T} = b - a$. Place the square of side a at the lower left corner of $[0, b] \times [0, \infty)$. Reserve the left strip of width a for a column of squares. To its right, place successive squares on *horizontal shelves*: rows of squares placed consecutively from left to right with their bottom edges aligned. Begin the first shelf at height zero. Whenever the next square does not fit in the available width $b - a$, put it in the left column immediately above the preceding column square and begin another shelf at the same height (Figure 2.1). Every square fits on an empty shelf, and every column square fits in the strip. Since the sides are decreasing and at most a, every square on a shelf is no taller than its column square, the first column square being the square of side a. Thus, the shelves do not overlap.

Let $u_1, u_2, \ldots$ be the sides of the column squares after the first, and w_j the total width occupied by the shelf immediately preceding u_j. Since u_j did not fit, $w_j + u_j > b - a$. Every square on that shelf has side at least u_j. Hence the total area of the shelf together with u_j^2 exceeds $u_j(b - a)$. These groups of squares are disjoint. Therefore, for each m for which u_m exists,

$$(b - a) \sum_{j=1}^{m} u_j \leq T.$$

Whether the number of shelves is finite or infinite, their total height is consequently at most

$$a + \sum_j u_j \leq a + \frac{T}{b - a} = b.$$

Every square is placed at a finite stage, and all shelves lie within the width $b - a$ and below height b. Therefore, all the squares fit in $[0, b]^2$.

For the last statement, assume $A > 0$. Since only finitely many sides exceed any positive number, there is a largest side a. The first part, applied to the remaining squares with $T = A - a^2$, gives a packing in a square of side $a + \sqrt{A - a^2} \leq \sqrt{2A}$, where the inequality follows from $2A - (a + \sqrt{A - a^2})^2 = (a - \sqrt{A - a^2})^2$. □

Lemma 2.2 (Existence, attainment, and truncation)**.** *A weakly decreasing nonnegative sequence admits a packing in some finite square if and only if it belongs to ℓ_2. For $s \in \ell_2^{\downarrow}$, the infimum defining $L^*(s)$ is attained. The function is monotone under coordinatewise increase, positively homogeneous, that is, $L^*(\lambda s) = \lambda L^*(s)$ for all $\lambda \geq 0$, and satisfies*

$$\|s\|_2 \leq L^*(s) \leq \sqrt{2}\, \|s\|_2, \tag{2.3}$$

$$\lim_{n \to \infty} L_n^*(s) = \sup_n L_n^*(s) = L^*(s). \tag{2.4}$$

Moreover, for all $n \geq 1$,

$$0 \leq L^*(s) - L_n^*(s) \leq \left(\sum_{i=n}^{\infty} s_i^2 \right)^{1/2}. \tag{2.5}$$

Proof. If s has a packing in $[0, L]^2$, countable additivity of area gives $\sum_i s_i^2 \leq L^2$, since the square interiors are disjoint. Hence $s \in \ell_2$, and taking the infimum over L gives $\|s\|_2 \leq L^*(s)$. Conversely, for every $s \in \ell_2^{\downarrow}$, Lemma 2.1 gives a packing in a square of side $\sqrt{2}\|s\|_2$. This proves the first assertion and (2.3).

If $s_i \leq t_i$ for every i, shrinking the squares in a packing of t while keeping their lower left corners fixed gives a packing of s in the same container. Thus, $L^*(s) \leq L^*(t)$. In particular, $L_n^*(s)$ is nondecreasing in n and bounded above by $L^*(s)$. Dilating packings gives $L^*(\lambda s) \leq \lambda L^*(s)$ for $\lambda > 0$, and applying the inverse dilation gives equality. The case $\lambda = 0$ follows from $L^*(0) = 0$.

Let $B = \sup_n L_n^*(s)$, which is finite by (2.3). For each $n \geq 1$, choose a packing of the positive entries among $s_0, \ldots, s_{n-1}$ in $[0, B + 1/n]^2$. A diagonal argument gives a subsequence along which the coordinates of every square with $s_i > 0$ converge. The conditions (2.1) and (2.2) are closed in the coordinates and the container side, so the limiting coordinates give a packing of s in $[0, B]^2$. Hence $L^*(s) \leq B$. Since $B \leq L^*(s)$, equality holds and the infimum is attained. Monotonicity of $L_n^*(s)$ then gives (2.4).

Finally, let $a = L_n^*(s)$. Since $s_i \leq s_{n-1} \leq a$ for $i \geq n$, Lemma 2.1 gives a packing of a square of side a together with squares of sides $s_n, s_{n+1}, \ldots$ in a square of side $a + (\sum_{i \geq n} s_i^2)^{1/2}$. Placing an optimal packing of $s_0, \ldots, s_{n-1}$ inside the square of side a proves the upper inequality in (2.5). The lower inequality follows from monotonicity. □

2.2. **Packing strategies and continuity.** The combinatorial framework below is closely related to the sequence-pair and constraint-graph method for fixed-orientation rectangles in [MFNK96].

Definition 2.3 (Strategies and complete families)**.** A *strategy* for n squares is a pair $\sigma = (\prec_{\mathrm{h}}, \prec_{\mathrm{v}})$ of strict partial orders on $\{0, \ldots, n-1\}$ such that every pair of distinct indices is comparable in at least one order. The relations prescribe horizontal and vertical separation. A *chain* of σ is a nonempty subset of $\{0, ..., n-1\}$ totally ordered by either relation. Write $\mathcal{C}(\sigma)$ for the set of all chains of σ and define the *cost* of σ at a vector $v \in [0, \infty)^n$ by $\ell_\sigma(v) = \max_{I \in \mathcal{C}(\sigma)} \sum_{i \in I} v_i$. For $s \in \ell_2^{\downarrow}$, we write $\ell_\sigma(s)$ for its value at $(s_0, \ldots, s_{n-1})$. A family $\mathcal{S}$ of strategies is *complete* if $L_n^*(s) = \min_{\sigma \in \mathcal{S}} \ell_\sigma(s)$ for every $s \in \ell_2^{\downarrow}$.

For the next lemma, let $t = (t_0, \ldots, t_{n-1})$ be a vector of nonnegative side lengths that need not be in decreasing order; we reserve s for nonincreasing sequences. A packing of t has squares $Q_i = [x_i, x_i + t_i] \times [y_i, y_i + t_i]$ for $t_i > 0$.

A packing of t *follows* a strategy σ if, for all i, j with $t_i, t_j > 0$, we have $x_i + t_i \leq x_j$ whenever $i \prec_{\mathrm{h}} j$ and $y_i + t_i \leq y_j$ whenever $i \prec_{\mathrm{v}} j$.

If $t_0, \ldots, t_{n-1} > 0$, then a packing of t determines a strategy σ, which we call the *strategy of the packing*: put $i \prec_{\mathrm{h}} j$ when $x_i + t_i \le x_j$, and $i \prec_{\mathrm{v}} j$ when $y_i + t_i \le y_j$. Positivity makes each relation irreflexive, the inequalities make it transitive, and (2.2) makes every pair comparable in at least one of them. The packing follows σ.

Lemma 2.4 (Criterion to pack following a strategy)**.** *Let σ be a strategy for n squares and let $t_0, \ldots, t_{n-1}$ be nonnegative, not necessarily in decreasing order. There is a packing of t in $[0, L]^2$ following σ if and only if $L \ge \ell_\sigma(t)$.*

Proof. Let I be a chain of $\prec_{\mathrm{h}}$, and list its members with positive sides in increasing order. In a following packing, adding the separation inequalities for consecutive members and the containment inequalities gives $\sum_{i \in I} t_i \le L$. The vertical argument is identical.

Conversely, for each j let x_j be the largest sum $\sum_{i \in I \setminus \{j\}} t_i$ over chains I of $\prec_{\mathrm{h}}$ with greatest element j, and define y_j using chains of $\prec_{\mathrm{v}}$. Singleton chains give $x_j, y_j \ge 0$, and $x_j + t_j, y_j + t_j \le \ell_\sigma(t)$. If $i \prec_{\mathrm{h}} j$, extend a maximizing chain ending at i by j to obtain $x_j \ge x_i + t_i$. Likewise $y_j \ge y_i + t_i$ for $i \prec_{\mathrm{v}} j$. Since every pair is comparable in at least one order, these coordinates give the desired packing. □

Proposition 2.5 (Strategy representation)**.** *For every $n \ge 1$ and every $s \in \ell_2^{\downarrow}$,*

$$L_n^*(s) = \min_{\sigma \in \mathfrak{S}_n} \ell_\sigma(s) = \min_{\sigma \in \mathfrak{S}_n} \max_{I \in \mathcal{C}(\sigma)} \sum_{i \in I} s_i, \tag{2.6}$$

where $\mathfrak{S}_n$ is the finite set of all strategies on n squares.

Proof. For every $\sigma \in \mathfrak{S}_n$, Lemma 2.4 gives a packing of $s_0, \ldots, s_{n-1}$ in a square of side $\ell_\sigma(s)$. Thus, $L_n^*(s) \le \min_{\sigma \in \mathfrak{S}_n} \ell_\sigma(s)$. Conversely, choose an optimal packing of the positive entries among $s_0, \ldots, s_{n-1}$, and take the horizontal and vertical separation orders on their labels. Extend the horizontal order by placing the labels of zero entries, in increasing index order, after all existing labels. Leave these added labels incomparable in the vertical order. The resulting pair of orders is a strategy σ followed by the packing. By Lemma 2.4, $\ell_\sigma(s) \le L_n^*(s)$, proving (2.6). □

We remark that each strategy cost is a maximum of finitely many linear subset sums, and there are only finitely many strategies. The formula therefore shows that L_n^* is continuous, positively homogeneous, and piecewise linear. At each input, the formula selects one of its constituent subset sums.

Lemma 2.6 (Continuity)**.** *For every $n \ge 1$ and $s, t \in \ell_2^{\downarrow}$,*

$$|L_n^*(s) - L_n^*(t)| \le \sum_{i < n} |s_i - t_i| \le \sqrt{n} \|s - t\|_2. \tag{2.7}$$

The function L^* is continuous on $\ell_2^{\downarrow}$ in the ℓ_2 norm. If $s - t \in \ell_1$, then

$$|L^*(s) - L^*(t)| \le \|s - t\|_1. \tag{2.8}$$

Proof. Proposition 2.5 expresses $L_n^*(s)$ as a finite minimum of maxima of subset sums of $s_0, \dots, s_{n-1}$. Each subset sum is 1-Lipschitz in the ℓ_1 norm; finite maxima and minima preserve a common Lipschitz constant. This proves (2.7).

For $u \in \ell_2^\downarrow$ and $n \geq 1$, write $u^{[n]} = (u_0, \dots, u_{n-1}, 0, 0, \dots)$ for the truncation of u, so that $L_n^*(u) = L^*(u^{[n]})$ and, by (2.5), $|L^*(u) - L_n^*(u)| \leq \|u - u^{[n]}\|_2$. Fix $s \in \ell_2^\downarrow$. For any t in that cone and any n,

$$\begin{aligned} |L^*(s) - L^*(t)| &\leq \sqrt{n}\|s - t\|_2 + \|s - s^{[n]}\|_2 + \|t - t^{[n]}\|_2 \\ &\leq (\sqrt{n} + 1)\|s - t\|_2 + 2\|s - s^{[n]}\|_2. \end{aligned}$$

First choose n to make the final term small, and then choose t close to s. This proves continuity. Finally, (2.7) gives $|L_n^*(s) - L_n^*(t)| \leq \|s - t\|_1$ for every n. Letting $n \to \infty$ and using (2.4) proves (2.8). □

The arguments below concern packings of actual squares, so they assume that the relevant sides are positive. The next lemma shows that lower bounds and formulas proved under this assumption hold in general.

Lemma 2.7 (Reduction to positive sides)**.** *Let $n \geq 1$, and let f be a continuous real-valued function of $(s_0, \dots, s_{n-1})$.*

(i) *If $L_n^*(s) \geq f(s)$ for every $s \in \ell_2^\downarrow$ with $s_{n-1} > 0$, then $L^*(s) \geq L_n^*(s) \geq f(s)$ for all $s \in \ell_2^\downarrow$.*

(ii) *If $L_n^*(s) = f(s)$ for every $s \in \ell_2^\downarrow$ with $s_{n-1} > 0$, then $L_n^*(s) = f(s)$ for all $s \in \ell_2^\downarrow$.*

Proof. Fix $s \in \ell_2^\downarrow$. For $\varepsilon > 0$, let s^ε be obtained from s by adding ε to each of $s_0, \dots, s_{n-1}$. Then $s^\varepsilon \in \ell_2^\downarrow$, $s_{n-1}^\varepsilon > 0$, and $s^\varepsilon \to s$ in ℓ_2 as $\varepsilon \to 0$. By continuity of L_n^* and f, letting $\varepsilon \to 0$ in the inequality or equality at s^ε gives it at s. In case (i), $L^*(s) \geq L_n^*(s)$ by monotonicity. □

3. Packing up to six squares

3.1. **Elementary chain estimates.** Fix a packing of squares Q_i of sides s_i in $[0, L]^2$. For a set of indices I, call $\sum_{i \in I} s_i$ its *side sum*; the same term applies to the corresponding collection $\{Q_i : i \in I\}$. We say that Q_i lies *left of* Q_j when $x_i + s_i \leq x_j$, and *below* Q_j when $y_i + s_i \leq y_j$. Two squares are *x-disjoint* when one lies left of the other, that is, when the interiors of their horizontal projections are disjoint, and *x-overlapping* otherwise. Define y-disjointness and y-overlap using vertical projections. By (2.2), every pair of squares in a packing is x-disjoint or y-disjoint (possibly both). A *horizontal chain* is a collection of pairwise x-disjoint squares, and a *vertical chain* is a collection of pairwise y-disjoint squares. For positive side lengths, the nonempty horizontal and vertical chains of a packing are exactly the chains of the strategy of the packing defined in Section 2.2.

Lemma 3.1 (Chain inequality)**.** *In a packing of side L, every horizontal or vertical chain has side sum at most L. In particular, $L^*(s) \geq s_0 + s_1$ for every $s \in \ell_2^\downarrow$.*

Proof. The projection intervals of a chain have disjoint interiors in $[0, L]$. Hence the sum of their lengths is at most L. Any two positive squares form a chain on at least one axis. If $s_1 = 0$, then $s_0 + s_1 = s_0 \le L$ by (2.1). □

3.2. **Four and five squares.** To check explicit coordinates, we use the elementary coefficient test for nonincreasing side lengths. Set $s_n = 0$. Then

$$\sum_{i=0}^{n-1} c_i s_i = \sum_{k=0}^{n-1} \left(\sum_{i=0}^{k} c_i \right) (s_k - s_{k+1}). \tag{3.1}$$

Thus, a linear form is nonnegative on every decreasing nonnegative tuple if and only if all its coefficient prefix sums are nonnegative. Necessity follows by taking the first $k+1$ sides equal to one and the rest zero. More generally, each coordinate containment or separation inequality below has the form $aL + \sum_{i=0}^{n-1} c_i s_i \ge 0$ with $a \ge 0$. When L is bounded below by the listed chain sums, we verify the inequality by replacing L with one of those lower bounds and applying the coefficient test.

Proposition 3.2. *For every $s \in \ell_2^{\downarrow}$,*

$$L_4^*(s) = s_0 + s_1, \tag{3.2}$$

$$L_5^*(s) = \max(s_0 + s_1, s_2 + s_3 + s_4). \tag{3.3}$$

In particular, $L_1^*(s) = s_0$ and $L_2^*(s) = L_3^*(s) = s_0 + s_1$.

Proof. By Lemma 2.7, it suffices to prove (3.2) when $s_3 > 0$ and (3.3) when $s_4 > 0$. Lemma 3.1 gives $L_4^*(s) \ge s_0 + s_1$. For $L \ge s_0 + s_1$, take lower left corners $(0, 0)$, $(s_0, 0)$, $(0, L - s_2)$, and $(L - s_3, L - s_3)$. Squares Q_2, Q_3 lie above Q_0, Q_1, respectively, Q_2 lies left of Q_1 and Q_3, and Q_0 lies left of Q_1 and Q_3. These relations and containment in $[0, L]^2$ follow from $L \ge s_0 + s_1$ by the coefficient test (3.1). Thus, $L_4^*(s) \le s_0 + s_1$. Since $L_n^*(s) = L_4^*(s_0, \ldots, s_{n-1}, 0, 0, \ldots)$ for $n \le 3$, the formulas for L_1^*, L_2^*, L_3^* are special cases of (3.2).

For five squares and $L \ge \max(s_0+s_1, s_2+s_3+s_4)$, take lower left corners $(0, 0)$, $(L-s_1, 0)$, $(0, L - s_2)$, $(s_2, L - s_3)$, and $(L - s_4, L - s_4)$ as in the second panel of Figure 1.1. Squares Q_2, Q_3, Q_4 form a horizontal chain along the top edge and lie above Q_0 and Q_1, while Q_0 lies left of Q_1. These relations and containment in $[0, L]^2$ follow from the two lower bounds for L by the coefficient test. Thus, $L_5^*(s) \le \max(s_0 + s_1, s_2 + s_3 + s_4)$. For the reverse inequality, it suffices to show that a packing of five squares contains a horizontal or vertical chain of three squares, since its side sum is at least $s_2 + s_3 + s_4$. Suppose no three of the squares are pairwise x-disjoint. Let A be the set of squares with no square to their left, and let B consist of the remaining squares. Two squares in A are not x-disjoint, since otherwise one would lie left of the other. Two squares $Q_i, Q_j \in B$ are not x-disjoint either: if Q_i lay left of Q_j, then a square Q_k left of Q_i would make Q_k, Q_i, Q_j pairwise x-disjoint. Thus, the squares in each of A and B are pairwise x-overlapping, hence pairwise y-disjoint. One of A, B contains at least three of the five squares, and these form a vertical chain. □

3.3. **The six-square construction.** Our coordinate formulas will satisfy a fixed choice of horizontal or vertical separation for every pair throughout the parameter range. The next lemma records how these uniform separations determine a strategy and its cost.

Lemma 3.3 (Uniform coordinates with separations define a strategy)**.** *Let g and x_i, y_i, $0 \le i < n$, be real-valued functions of $(s_0, \dots, s_{n-1})$, with $g \ge 0$. Suppose that the coordinates $(x_i(s), y_i(s))$ pack $s_0, \dots, s_{n-1}$ in $[0, g(s)]^2$ for every $s \in \ell_2^{\downarrow}$, with each pair satisfying a fixed separation inequality from (2.2) for all s. Define $i \prec_{\mathrm{h}} j$ if $x_i(s) + s_i \le x_j(s)$ for all s, and $i \prec_{\mathrm{v}} j$ if $y_i(s) + s_i \le y_j(s)$ for all s. Then $\sigma = (\prec_{\mathrm{h}}, \prec_{\mathrm{v}})$ is a strategy with $\ell_\sigma \le g$. If g is the maximum of the side sums of a fixed collection of chains of σ, then $\ell_\sigma = g$.*

Proof. The relations are transitive, and evaluating at positive side vectors shows that they are irreflexive. The hypothesis on pairs makes σ a strategy, and the given packings follow σ. Thus, Lemma 2.4 gives $\ell_\sigma \le g$. The final assertion follows from the definition of ℓ_σ. □

We now give a single strategy for six squares that works for every nonincreasing side vector.

Proposition 3.4 (Six-square construction)**.** *For every $s \in \ell_2^{\downarrow}$,*

$$L_6^*(s) \le \max(s_0 + s_1,\ s_2 + s_3 + s_4,\ s_1 + s_4 + s_5).$$

More precisely, for L at least the right-hand side, the coordinates

(3.4)

i	0	1	2
(x_i, y_i)	$(0, 0)$	$(L - s_1, 0)$	$(0, L - s_2)$
i	3	4	5
(x_i, y_i)	$(s_2, L - s_3)$	$(L - s_4, L - s_4)$	$(L - s_5, s_1)$

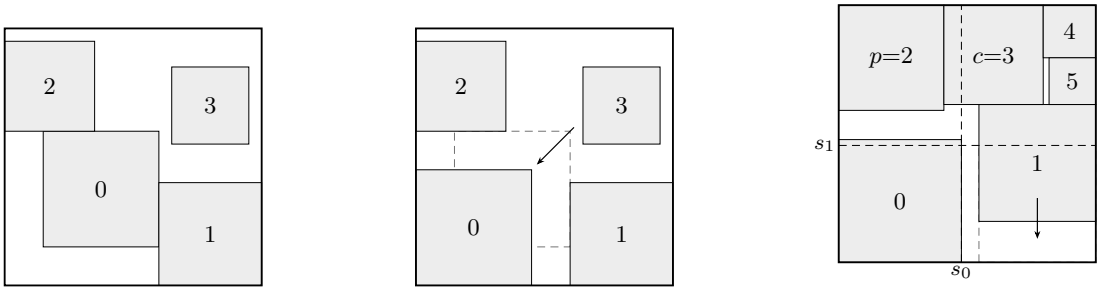
pack $Q_0, \dots, Q_5$ in $[0, L]^2$. Setting L equal to the displayed maximum defines a strategy whose cost is that maximum.

Proof. Let $g(s)$ be the displayed maximum. For every $L \ge g(s)$, the coefficient test (3.1) verifies that the squares lie in $[0, L]^2$ and that each pair satisfies a fixed separation inequality. The collections $\{Q_0, Q_1\}$ and $\{Q_2, Q_3, Q_4\}$ are horizontal chains, while Q_1, Q_5, Q_4 form a vertical chain in that order. Setting $L = g(s)$ and applying Lemma 3.3 therefore gives a strategy with cost $g(s)$. Proposition 2.5 gives the upper bound for L_6^*. □

Figure 1.1 illustrates the constructions.

3.4. **Four large squares.** The five-square lower bound needed only the existence of a three-square chain. For six squares we need more information, and we first set up the method used for all remaining lower bounds.

Suppose that we wish to prove a lower bound of the form $L^*(s) \ge \Sigma$. We assume to the contrary that some packing fits in a square of side $L < \Sigma$. We say that a set of indices I, or the corresponding collection of squares $\{Q_i : i \in I\}$, *clears* the threshold Σ when $\sum_{i \in I} s_i \ge \Sigma$. By Lemma 3.1, such a collection cannot be a horizontal or vertical chain.



(a) windowpane, $L = 20$
$s = (9, 8, 7, 6)$

(b) cornering, $L = 20$
$s = (9, 8, 7, 6)$

(c) forced relations, $L = 44$
$s = (21, 20, 18, 17, 9, 8)$

FIGURE 3.1. (a) A windowpane in a square of side $20 < s_1 + s_2 + s_3$: the columns are $\{Q_0, Q_2\}$ and $\{Q_1, Q_3\}$, and the rows are $\{Q_0, Q_1\}$ and $\{Q_2, Q_3\}$. (b) The cornering step of Lemma 3.5: Q_0 has no blocker to its left or below, and moves left and then down from the dashed position to the lower left corner. (c) The conclusions of Lemma 3.6 with $r = N = 5$, where every triple in the hypotheses has side sum at least $s_1 + s_3 + s_5 = 45 > L$. Here $p = 2$ and $c = 3$; the squares Q_4, Q_5 are x-disjoint from Q_p, Q_c, y-overlap both, and lie above height s_1. Square Q_1 can be lowered to the floor. The dashed lines are $x = s_0$ and $y = s_1$, and the dashed outline is the lowered position of Q_1.

The proofs below use these exclusions to force overlaps and then analyze the resulting geometry to get a contradiction.

For example, suppose Q_a, Q_b are y-disjoint and $\{a, b, e\}$ clears Σ. Then Q_e must y-overlap Q_a or Q_b, since otherwise the three squares would form a vertical chain with side sum exceeding L. Whenever two squares y-overlap, they must be x-disjoint. We call the forced y-overlap with one of Q_a, Q_b a *vertical pin* of Q_e to the pair Q_a, Q_b. A pin to a longer chain, or a horizontal pin, means the analogous necessary overlap.

We will sometimes translate a square to a wall while keeping the other squares fixed. For instance, a *blocker to the right* of Q_i is a square to its right whose vertical projection overlaps that of Q_i: moving Q_i right would eventually meet it. If there is no such blocker, Q_i can be moved to the right wall without disturbing the packing. Blockers in other directions are defined analogously. Reflections and interchanges of the axes move the entire packing. Because clearing depends only on the side lengths, an exclusion based on a clearing set remains valid during any motion that preserves a packing.

If every triple among the four largest squares clears Σ, their projections must have a constrained arrangement, which we call a *windowpane* (Figure 3.1(a)). Its "columns" will be x-overlapping pairs and its "rows" y-overlapping pairs. With an additional clearing hypothesis, this structure also lets us move the largest square into a corner of the full packing (Figure 3.1(b)). Both facts will be used again for seven squares.

Lemma 3.5 (Windowpane and cornering)**.** *Fix a packing of side $L < \Sigma$, and suppose every triple among Q_0, Q_1, Q_2, Q_3 clears Σ. The four squares split into two x-overlapping pairs, called columns, and into two y-overlapping pairs, called rows. No square x-overlaps both members of the other column, or y-overlaps both members of the other row.*

If, in addition, every triple $\{0, b, j\}$ with $b \in \{1, 2, 3\}$ and $j \notin \{0, b\}$ in the full packing clears Σ, then Q_0 can be translated to a corner in that full packing.

Proof. Make a graph with vertices $0, 1, 2, 3$, joining two indices when their squares are x-disjoint. The graph has no triangle, since three pairwise x-disjoint squares would form a clearing horizontal chain. Since it has only four vertices, it has no odd cycle and is bipartite. Each of its two parts consists of pairwise x-overlapping squares. Such squares are pairwise y-disjoint, and a part cannot contain three indices, since a vertical triple would clear Σ. Hence the parts both have size two, giving the two columns. Repeating the argument with the coordinate axes interchanged gives the two rows. A square that x-overlapped both members of the other column would form a forbidden vertical triple with them. The row assertion follows by symmetry.

For the cornering statement, consider blockers on opposite sides of Q_0 in the horizontal direction. They are x-disjoint from Q_0 and from each other. Together with Q_0 they form a horizontal chain. If either blocker is among Q_1, Q_2, Q_3, this triple clears Σ. If both are smaller squares, there is a square Q_b, $b \in \{1, 2, 3\}$, x-disjoint from Q_0: otherwise Q_0 and both squares of the other column would form a forbidden vertical triple. Square Q_b is on one side of Q_0 and, together with the blocker on the opposite side, forms a clearing horizontal triple with Q_0. Thus, there cannot be blockers on both sides. Translate Q_0 to a vertical wall, and repeat the argument vertically to reach a corner; reflect the packing if needed. All the side-sum exclusions remain valid during these motions. □

The next lemma concerns the $r+1$ largest squares $Q_0, \dots, Q_r$ of a packing of $Q_0, \dots, Q_N$, and its hypotheses involve only indices in $\{0, \dots, r\}$. In the proof we remove $Q_{r+1}, \dots, Q_N$, lower Q_1 in the resulting packing of $Q_0, \dots, Q_r$, derive relations among squares whose positions have not changed, and then use those relations in the original packing. The auxiliary translation need not be possible when $Q_{r+1}, \dots, Q_N$ are restored. Figure 3.1(c) illustrates the conclusions.

Lemma 3.6 (Relations forced by four large squares)**.** *Suppose $Q_0 = [0, s_0]^2$ in a packing of $Q_0, \dots, Q_N$ of side $L < \Sigma$, where $N \geq r \geq 4$. Assume that the following triples of indices in $\{0, \dots, r\}$ clear Σ:*

(i) *every triple among $0, 1, 2, 3$;*
(ii) *$\{0, 1, j\}$ for $2 \leq j \leq r$;*
(iii) *$\{0, b, m\}$ and $\{1, b, m\}$ for $b \in \{2, 3\}$ and $4 \leq m \leq r$.*

After possibly interchanging axes, $x_1 \geq s_0$ and $y_1 < s_0$. There are distinct $p, c \in \{2, 3\}$ such that p x-overlaps 0, c x-overlaps 1, and c lies above 1. Moreover, in the original full packing:

(a) *p, c are x-disjoint, with p left of c;*

(b) *each* $m \in \{4, \dots, r\}$ *is* x*-disjoint from both* p, c *and* y*-overlaps both of them;*
(c) $y_m \geq s_1$ *for every such* m*.*

If $r = N$*, square* Q_1 *can be moved to the floor in the full packing.*

Proof. Apply the windowpane statement and choose axes so that $x_1 \geq s_0$. Suppose first that $y_1 \geq s_0$. Let p and q be the column and row partners of 0; they are $2, 3$ in some order. Then p lies above 0, q lies right of 0, and 1 lies right of p and above q. These relations follow from the windowpane, since $1, p$ lie in a row and $1, q$ in a column, together with $x_p < s_0 \leq x_1$ and $y_q < s_0 \leq y_1$.

Set $m = 4$. The vertical pin of Q_m to Q_0, Q_1 forces it to y-overlap one of them. If it overlaps 0, it lies right of 0, and the vertical pin to Q_1, Q_q forces an overlap with 1 or q. It is x-disjoint from that square, and both are right of 0, giving a clearing horizontal triple. Otherwise m is y-disjoint from 0 and y-overlaps 1. It is x-disjoint from 1; the horizontal pin to Q_p, Q_1 makes it x-overlap Q_p. The vertical chain Q_0, Q_p, Q_m now clears Σ, a contradiction. Hence $y_1 < s_0$.

Let c be the column partner of 1 and p the column partner of 0. Suppose c lies below 1. Then $y_c + s_c \leq y_1 < s_0$. Hence c lies right of 0, whereas p lies above 0. Square p cannot lie below 1, and cannot lie above it because Q_c, Q_1, Q_p would be a vertical chain. Thus, p y-overlaps 1 and is x-disjoint from it.

Again take $m = 4$. The vertical pin of Q_m to Q_c, Q_1 forces a y-overlap. If it overlaps c but not 1, it lies below 1 and right of 0, and is x-disjoint from c, giving the clearing horizontal chain Q_0, Q_c, Q_m. Therefore, Q_m y-overlaps Q_1 and is x-disjoint from it. Hence the horizontal pin to Q_0, Q_1 forces Q_m to x-overlap Q_0. Thus, Q_m is y-disjoint from $Q_0 = [0, s_0]^2$ and lies above it. If m, p are x-disjoint, the horizontal chain Q_1, Q_p, Q_m clears Σ; otherwise the vertical chain Q_0, Q_p, Q_m clears Σ. Hence c lies above 1.

Now remove $Q_{r+1}, \dots, Q_N$. In the packing of $Q_0, \dots, Q_r$, square 1 can be lowered to the floor with all the other squares fixed. A downward blocker other than 0 would complete a clearing vertical triple with $1, c$, while 0 is x-disjoint from 1. In the auxiliary placement, Q_0 and the lowered Q_1 both lie on the floor. Each of $Q_p, Q_4, \dots, Q_r$ is pinned horizontally to Q_0, Q_1. It therefore x-overlaps one of them, is y-disjoint from it, and lies above it. Since $s_0 \geq s_1$, each of these squares lies above height s_1. The vertical pins to Q_1, Q_c make these squares y-overlap Q_c, hence be x-disjoint from it. The pins to Q_1, Q_p likewise make each $m \geq 4$ y-overlap p, hence be x-disjoint from it.

All coordinates occurring in (a)–(c) have remained fixed; only the vertical coordinate of 1 changed. Thus, these assertions hold in the original full packing, regardless of $Q_{r+1}, \dots, Q_N$. Finally, p lies left of c: the reverse order would give $x_c + s_c \leq x_p < s_0 \leq x_1$, contradicting the x-overlap of $c, 1$. If $r = N$, the auxiliary motion is a motion of the full packing. □

3.5. **A column bound and the six-square formula.**

Proposition 3.7 (Column bound)**.** *For every $n \geq 6$ and $s \in \ell_2^{\downarrow}$,*

$$L^*(s) \geq L_n^*(s) \geq \min\left(s_1 + s_3 + s_{n-1},\ P_n(s)\right), \qquad \text{where } P_n(s) = s_1 + \sum_{i=4}^{n-1} s_i. \tag{3.5}$$

Proof. By Lemma 2.7, it suffices to prove the inequality for L_n^* when $s_{n-1} > 0$. Set

$$D = s_1 + s_3 + s_{n-1} \quad \text{and} \quad Z = s_2 + s_3 + s_{n-2} + s_{n-1},$$

and suppose that $L < \Sigma := \min(D, P_n, Z)$. Every triple of indices among $0, 1, 2, 3$ clears Σ; so do $\{0, b, j\}$ with $b \in \{1, 2, 3\}$ and any other index j, and $\{1, b, m\}$ with $b \in \{2, 3\}$ and $m \geq 4$. Each of these triples has side sum at least D, because s is nonincreasing. By Lemma 3.5, we may place Q_0 in the lower left corner. Lemma 3.6, with $r = N = n - 1$, then allows us, after possibly interchanging axes, to lower Q_1 to the floor. It also shows that Q_2, Q_3 are x-disjoint from one another and from every square with index at least 4, all of which lie above Q_1.

If two of the squares $Q_4, \ldots, Q_{n-1}$ are x-disjoint, they and Q_2, Q_3 form a horizontal chain of sum at least Z. Otherwise these squares are pairwise x-overlapping and form a vertical chain. With Q_1 below all of them, this chain has sum P_n. Either chain contradicts $L < \Sigma$. Thus, $L \geq \min(D, P_n, Z)$ in every packing.

It remains to show that $L_n^* \geq \min(D, P_n)$. If $\min(D, P_n) \leq s_0 + s_1$, use Lemma 3.1. Otherwise $D > s_0 + s_1$, which gives $s_3 + s_{n-1} > s_0 \geq s_1$ and $s_2 + s_{n-2} \geq s_3 + s_{n-1} > s_1$. It follows that $Z > D$, proving (3.5). □

Theorem 3.8. *For every $s \in \ell_2^{\downarrow}$,*

$$L_6^*(s) = \max(s_0 + s_1,\ s_2 + s_3 + s_4,\ s_1 + s_4 + s_5). \tag{3.6}$$

Proof. Proposition 3.4 proves the upper bound. Lemma 3.1 gives $L_6^* \geq s_0 + s_1$, and $L_6^* \geq L_5^* \geq s_2 + s_3 + s_4$ by (3.3). Proposition 3.7 with $n = 6$ gives the remaining lower bound $L_6^* \geq \min(s_1 + s_3 + s_5,\ s_1 + s_4 + s_5) = s_1 + s_4 + s_5$, because $s_3 \geq s_4$. This proves (3.6). □

4. Sharp bounds on the number of sides in a sum

Proposition 2.5 guarantees that every value $L_n^*(s)$ is some subset sum. We now determine how many sides that sum may need. For $s \in \ell_2^{\downarrow}$, call $I \subseteq \{0, \ldots, n-1\}$ a *representing set* if $L_n^*(s) = \sum_{i \in I} s_i$, and put $q_n(s) = \min\{|I| : I \text{ represents } s\}$. We prove $\max_{s \in \ell_2^{\downarrow}} q_n(s) = n - 3$ for $n \geq 6$.

4.1. Representing sums with at most $n - 3$ sides.

We begin with two placements that work for any number of squares.

Proposition 4.1 (Two general constructions)**.** *For $n \geq 6$ and P_n as in Proposition 3.7, one has*

$$L_n^*(s) \leq \max(s_0 + s_1,\ s_2 + s_3 + s_4,\ P_n). \tag{4.1}$$

Also,

$$L_n^*(s) \le \max\left(s_0 + s_1,\ s_0 + s_2 + s_3,\ \sum_{\substack{2 \le i < n \\ i \text{ even}}} s_i\right).$$

In particular, if $s_2 + s_3 \le s_1$*, then*

$$L_n^*(s) \le \max\left(s_0 + s_1,\ \sum_{\substack{2 \le i < n \\ i \text{ even}}} s_i\right). \tag{4.2}$$

Proof. For the first construction, set $c = \max(s_0, s_2 + s_3)$ and place $Q_0, \dots, Q_3$ with respective lower left corners

$$(0,0), \quad (s_0,0), \quad (0,s_0), \quad (s_2,s_0).$$

For $4 \le i \le n-1$, put $(x_i, y_i) = (c,\ s_1 + \sum_{j=4}^{i-1} s_j)$. The column squares are mutually y-disjoint, lie above Q_1, and lie right of Q_0, Q_2, Q_3. The shelf squares Q_2, Q_3 are x-disjoint and lie above the two bottom squares. The horizontal extent is $\max(s_0 + s_1, c + s_4) = \max(s_0 + s_1, s_2 + s_3 + s_4)$, since $s_4 \le s_1$. The vertical extent is $\max(s_0 + s_2, P_n) \le \max(s_0 + s_1, P_n)$.

For the second construction, retain Q_0, Q_1 at the bottom. Place the even-indexed squares $Q_2, Q_4, \dots$ consecutively from left to right on the shelf at height s_0, and the odd-indexed squares $Q_3, Q_5, \dots$ on the shelf at height $s_0 + s_2$. Their heights are at most s_2 and s_3, respectively. By termwise comparison, the odd-indexed shelf is no wider than the even-indexed one. The bounds follow from the horizontal and vertical extents. □

We next show that representing sets with at least $n-2$ elements can be avoided.

Theorem 4.2 (Representing sums with at most $n-3$ sides)**.** *Let* $n \ge 6$ *and* $s \in \ell_2^{\downarrow}$ *with* $s_{n-1} > 0$*. If a representing set* I *has* $|I| \ge n-2$*, then* $L_n^*(s) = s_0 + s_1$*. For every* $s \in \ell_2^{\downarrow}$*, including vectors with zero entries,* $q_n(s) \le n-3$*.*

Proof. Let $J = I^c$. Then $|J| \le 2$, so $\sum_{j \in J} s_j \le s_0 + s_1$ and

$$\sum_{i \in I} s_i \ge \sum_{i=2}^{n-1} s_i. \tag{4.3}$$

The chain inequality gives $\sum_{i \in I} s_i = L_n^* \ge s_0 + s_1$. Suppose it is strictly larger; we obtain a packing of side strictly less than $\sum_{i \in I} s_i$.

If $s_1 < s_2 + s_3$, use Proposition 4.1. Its first term is smaller than $\sum_{i \in I} s_i$ by assumption. Its second is smaller by (4.3), since the tail beginning with s_5 is nonempty and positive. For the third, both sums omit entries from the sum of all n sides, whence

$$\sum_{i \in I} s_i - P_n = (s_0 + s_2 + s_3) - \sum_{j \in J} s_j \ge s_2 + s_3 - s_1 > 0.$$

If $s_2 + s_3 \le s_1$, use (4.2). The first term is smaller than $\sum_{i \in I} s_i$. The even-indexed sum beginning with s_2 is strictly smaller than $\sum_{i=2}^{n-1} s_i \le \sum_{i \in I} s_i$, because the latter also includes $s_3 > 0$. This is again a contradiction.

Thus, for positive side lengths, every representing set of cardinality at least $n-2$ can be replaced by $\{0,1\}$, proving $q_n(s) \le n-3$. If some entries vanish, apply this result to the positive prefix when it has at least six entries, and otherwise use the formulas for at most five squares. □

The two constructions serve different parameter ranges. For example, with $n = 10$ and sides $(5, 5, 2, \dots, 2)$, the column construction gives an upper bound of 17, whereas the two-shelf construction gives the exact value 10.

4.2. **Sharpness and consequences.** We now show that the bound $n-3$ is attained on an open set of side vectors.

Theorem 4.3 (Sharpness)**.** *For every $n \ge 6$, there is a nonempty open set U of strictly decreasing positive n-tuples such that, for all $s \in \ell_2^{\downarrow}$ with $(s_0, \dots, s_{n-1}) \in U$,*

$$L_n^*(s) = P_n(s) = s_1 + \sum_{i=4}^{n-1} s_i \quad \textit{and} \quad q_n(s) = n-3.$$

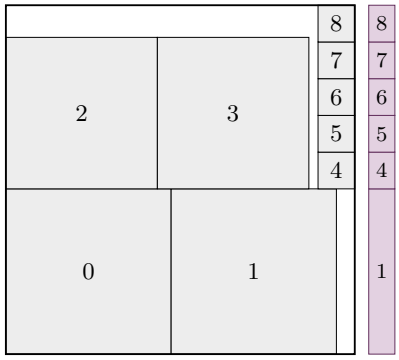
Proof. Let $n \ge 6$, set $m = n-5$, and consider

$$s_0 = s_1 = 8m+4, \qquad s_2 = s_3 = 8m+1, \qquad s_4 = \cdots = s_{n-1} = 8. \tag{4.4}$$

There are $m+1$ sides equal to 8. At this input, $P_n = 16m+12$ and

$$s_0 + s_1 = P_n - 4, \qquad s_2 + s_3 + s_4 = P_n - 2, \qquad s_1 + s_3 + s_{n-1} = P_n + 1.$$

The column construction and the column bound of Proposition 3.7 give $L_n^* = P_n$. Figure 4.1 shows the member of this family with $m = 4$. The three displayed comparisons



$$L_9^* = 76$$
$$s = (36, 36, 33, 33, 8, 8, 8, 8, 8)$$

FIGURE 4.1. The member of the sharpness family (4.4) with $m = 4$. The six-term vertical chain has side sum 76, whereas $s_0 + s_1 = 72$, $s_2 + s_3 + s_4 = 74$, and $s_1 + s_3 + s_8 = 77$. The lower bound proves that the column cost is optimal.

remain strict in a neighborhood of this vector. Intersecting that neighborhood with the cone of strictly decreasing positive n-tuples gives a nonempty open set on which $L_n^* = P_n$.

For each subset I of cardinality less than $n-3$, the linear forms P_n and $\sum_{i\in I} s_i$ have different coefficient vectors. Their equality defines a proper hyperplane. Since there are only finitely many such subsets, we may choose a smaller open ball that avoids all of these hyperplanes. On that ball, P_n has no representation using fewer than $n-3$ sides. □

Finally, we deduce consequences for linear components, for minimum-of-maxima formulas, and for Lipschitz constants. A *linear component* means a linear function of $(s_0, \dots, s_{n-1})$ that agrees with $L_n^*(s)$ on a nonempty open set of strictly decreasing positive n-tuples.

Corollary 4.4 (Minimum-of-maxima formulas with at most $n-3$ sides per sum)**.** *For $n \geq 6$, every linear component is a subset sum with at most $n-3$ entries. There is a finite minimum-of-maxima formula for $L_n^*(s)$ in $s_0, \dots, s_{n-1}$ whose constituent sums all have that cardinality bound. No such representation using only shorter subset sums exists.*

Proof. The finite strategy formula subdivides the cone of nonincreasing nonnegative n-tuples into finitely many polyhedral regions on which $L_n^*(s)$ is a subset sum. A linear component agrees with one of these subset sums on an open subset of its region, and hence is a subset sum. On a full-dimensional region, Theorem 4.2 excludes a component of cardinality at least $n-2$: it would equal $s_0 + s_1$ on an open set, hence as a linear form, contrary to its different coefficient vector. Boundary values follow by continuity from the full-dimensional regions.

The lattice representation theorem [Ovc02], applied to $-L_n^*$ on the closed convex cone of nonincreasing nonnegative n-tuples, expresses L_n^* as a finite minimum of maxima of its linear components. These are the short subset sums identified above. Finally, any finite minimum-of-maxima expression selects one of its constituent values at every input. Theorem 4.3 therefore rules out using only shorter subset sums. □

Corollary 4.5 (Sharp Euclidean Lipschitz constants)**.** *For every $n \geq 6$,*

$$|L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3}\,\|s-t\|_2 \quad \text{for all } s,t \in \ell_2^{\downarrow},$$

and the constant $\sqrt{n-3}$ cannot be decreased. In particular L^* is not globally Lipschitz on $\ell_2^{\downarrow}$ in the ℓ_2 metric.

Proof. A sum of at most $n-3$ coordinates has gradient of Euclidean norm at most $\sqrt{n-3}$. Since finite minima and maxima preserve a common Lipschitz constant, Corollary 4.4 gives

$$|L_n^*(s) - L_n^*(t)| \leq \sqrt{n-3}\Big(\sum_{i<n}(s_i - t_i)^2\Big)^{1/2} \leq \sqrt{n-3}\,\|s-t\|_2.$$

Now, by Theorem 4.3, $L_n^* = P_n$ on a nonempty open set of strictly decreasing positive n-tuples. Since $\|\nabla P_n\|_2 = \sqrt{n-3}$, the constant is sharp. On sequences obtained by extending n-tuples with zeros, $L^* = L_n^*$. Thus, a global Lipschitz constant for L^* would have to be at least $\sqrt{n-3}$ for every n, which is impossible. □

Together, Theorem 4.2, Theorem 4.3, and the two corollaries prove Theorem B.

5. Seven squares

5.1. **Three strategies.** Put

$$W = s_3 + s_4 + s_5 + s_6, \qquad H = s_1 + s_2 + s_5, \tag{5.1}$$
$$C_0 = s_0 + s_5 + s_6, \qquad C_3 = s_1 + s_3 + s_6, \qquad R = s_1 + s_4 + s_5 + s_6.$$

and

$$F_7 = \max\bigl(L_6^*, \min(\max(W, H), \max(C_0, C_3), R)\bigr). \tag{5.2}$$

We use the coordinates in Table 5.1 to construct three strategies. They are named for the position of Q_6: against the top edge, in the interior, and against the right edge.

Proposition 5.1 (Three seven-square strategies)**.** *Setting L in each row of Table 5.1 equal to the corresponding right-hand side below defines strategies $\sigma_{\text{top}}, \sigma_{\text{center}}, \sigma_{\text{right}}$ with costs*

$$\ell_{\text{top}} = \max(L_6^*, W, H), \quad \ell_{\text{center}} = \max(L_6^*, C_0, C_3), \quad \ell_{\text{right}} = \max(L_6^*, R).$$

For each strategy and every L at least its cost, the corresponding row of Table 5.1 gives a packing in $[0, L]^2$. Consequently, for every $s \in \ell_2^{\downarrow}$,

$$L_7^*(s) \le \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}) = F_7(s).$$

Proof. For each row, let $g(s)$ be the corresponding right-hand side in the statement. The coefficient test (3.1) verifies that the squares lie in $[0, L]^2$ and that each pair satisfies a fixed separation inequality whenever $L \ge g(s)$. Setting $L = g(s)$ and applying Lemma 3.3 gives a strategy with cost at most $g(s)$. For example, in the center strategy Q_6 lies right of Q_0 because $x_6 = L - s_5 - s_6 \ge s_0$ when $L \ge C_0$, and below Q_3 because $y_6 + s_6 = s_1 + s_6 \le L - s_3 = y_3$ when $L \ge C_3$. The following chain sums give the reverse inequality:

	horizontal	vertical
top	$s_0 + s_1, s_2 + s_3 + s_4, s_3 + s_4 + s_5 + s_6$	$s_1 + s_2 + s_5$
center	$s_0 + s_1, s_2 + s_3 + s_4, s_0 + s_5 + s_6$	$s_1 + s_4 + s_5, s_1 + s_3 + s_6$
right	$s_0 + s_1, s_2 + s_3 + s_4$	$s_1 + s_4 + s_5 + s_6$.

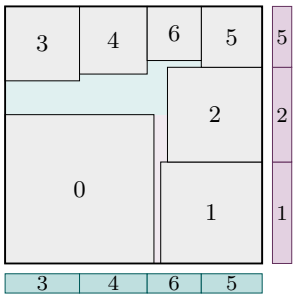
Each term of (3.6) that is not displayed is dominated by a displayed sum, which gives the three costs. Their minimum is F_7, giving the asserted upper bound for L_7^*. □

Figure 1.2 shows chains attaining the costs.

strategy	(x_0, y_0) (x_4, y_4)	(x_1, y_1) (x_5, y_5)	(x_2, y_2) (x_6, y_6)	(x_3, y_3)
top	$(0,\ 0)$ $(s_3,\ L - s_4)$	$(L - s_1,\ 0)$ $(L - s_5,\ L - s_5)$	$(L - s_2,\ s_1)$ $(s_3 + s_4,\ L - s_6)$	$(0,\ L - s_3)$
center	$(0,\ 0)$ $(L - s_4,\ L - s_4)$	$(L - s_1,\ 0)$ $(L - s_5,\ s_1)$	$(0,\ L - s_2)$ $(L - s_5 - s_6,\ s_1)$	$(s_2,\ L - s_3)$
right	$(0,\ 0)$ $(L - s_4,\ L - s_4)$	$(L - s_1,\ 0)$ $(L - s_5,\ s_1)$	$(0,\ L - s_2)$ $(L - s_6,\ s_1 + s_5)$	$(s_2,\ L - s_3)$

TABLE 5.1. Coordinates of the lower left corners of $Q_0, \ldots, Q_6$ for the three strategies top, center, and right, as functions of the tuple s and the side L.

5.2. **Cornering and a perpendicular chain cover.** For seven squares, a more general cornering argument replaces the four-square windowpane. When every triple containing the largest square clears Σ, we can move that square to a corner. Each remaining square must then lie to its right or above it. Those two groups form chains in perpendicular directions (Figure 5.1), and the following lemmas show how to select a pair of chains that accounts for all the remaining squares. This will force a four-square chain in the proof of Proposition 5.4.



corner frame, $L = 38$
$s = (22, 15, 14, 11, 10, 9, 8)$

FIGURE 5.1. The corner frame of Lemma 5.2 for the top strategy, where every triple through Q_0 has side sum at least $s_0 + s_5 + s_6 = 39 > L$. The right arm $\{1, 2, 5\}$ lies in the purple region $x \geq s_0$, the upper arm $\{3, 4, 5, 6\}$ lies in the teal region $y \geq s_0$, and the arms share Q_5.

Lemma 5.2 (Corner frame)**.** *Fix a packing of $Q_0, \ldots, Q_N$ of side $L < \Sigma$ and an index $m \leq N$, and suppose every triple of indices in $\{0, \ldots, m\}$ that contains 0 clears Σ. Then Q_0 can be translated to a corner in the packing of $Q_0, \ldots, Q_m$ obtained by removing $Q_{m+1}, \ldots, Q_N$. After reflection, let $Q_0 = [0, s_0]^2$ and put*

$$\mathcal{R} = \{i \in \{1, \ldots, m\} : x_i \geq s_0\}, \qquad \mathcal{U} = \{i \in \{1, \ldots, m\} : y_i \geq s_0\}.$$

The arms cover $\{1,\dots,m\}$. The right arm is a vertical chain with pairwise x-overlaps, the upper arm is a horizontal chain with pairwise y-overlaps, and $|\mathcal{R}\cap\mathcal{U}|\le 1$. If Q_0 is already cornered in the packing of $Q_0,\dots,Q_N$, these conclusions about $Q_1,\dots,Q_m$ require no motion.

Proof. Blockers on both sides of the horizontal band through Q_0 would form a clearing horizontal triple with it. One side is therefore free; move Q_0 to a vertical wall. Repeat vertically and reflect. Every other square then lies right of Q_0 or above it. Two x-disjoint members of $\mathcal{R}$ would form a clearing horizontal triple with Q_0. They therefore x-overlap and are y-disjoint. The upper-arm assertion is symmetric. Two members in the intersection would be required both to x-overlap, as right-arm members, and to be x-disjoint, as upper-arm members. □

Lemma 5.3 (Perpendicular covering chains)**.** *In the frame of Lemma 5.2, with $m\ge 1$, there are a vertical chain $\mathcal{V}$ and a horizontal chain $\mathcal{H}$ such that $\mathcal{V}\cup\mathcal{H}=\{1,\dots,m\}$ and $\mathcal{V}\cap\mathcal{H}$ consists of one index. In particular, for $m=6$ some four squares form a chain.*

Proof. If one arm is empty, the other arm is a chain containing every index, and we pair it with a singleton. If the arms meet, they are the required chains. Otherwise the arms are disjoint and nonempty. Let r be topmost in $\mathcal{R}$ and u rightmost in $\mathcal{U}$. Since

$$x_r \ge s_0 > x_u, \qquad y_u \ge s_0 > y_r,$$

either r lies right of u, or u lies above r. In the first case, since u is the rightmost member of $\mathcal{U}$ and lies left of r, every member of $\mathcal{U}$ lies left of r. Thus, $\mathcal{R}$ and $\mathcal{U}\cup\{r\}$ are the required chains. In the second case, $\mathcal{R}\cup\{u\}$ and $\mathcal{U}$ are the required chains by symmetry. The two chain cardinalities sum to $m+1$. Hence for $m=6$ one of them is at least four. □

5.3. **Proof of the seven-square formula.** Put $W_{24}=s_2+s_4+s_5+s_6$.

Proposition 5.4 (Three lower bounds)**.** *For every $s\in\ell_2^{\downarrow}$,*

$$L_7^*(s)\ge\min(C_0,W), \tag{5.3}$$

$$L_7^*(s)\ge\min(C_3,R), \tag{5.4}$$

$$L_7^*(s)\ge\min(C_0,H,W_{24}). \tag{5.5}$$

Proof. By Lemma 2.7, we may assume $s_6>0$. For (5.3), put $\Sigma=\min(C_0,W)$ and suppose a packing has side $L<\Sigma$. Every triple through Q_0 has side sum at least C_0 and clears Σ. Corner Q_0 and apply Lemma 5.3 to the six remaining squares. The resulting four-square chain has side sum at least W, a contradiction. Inequality (5.4) is Proposition 3.7 with $n=7$.

For (5.5), suppose $L<\Sigma:=\min(C_0,H,W_{24})$. Corner Q_0 and form the corner frame on $Q_1,\dots,Q_6$. Every four-subset of these indices except $\{3,4,5,6\}$ has side sum at least W_{24} and clears Σ. Thus, the four-chain supplied by Lemma 5.3 must be exactly $\{3,4,5,6\}$; interchange axes to make it horizontal. At most one of its members lies in the right arm.

Hence at least three lie in the upper arm. Neither Q_1 nor Q_2 can lie in the upper arm, because it and these three would form a clearing horizontal quadruple. Thus, Q_1, Q_2 lie in the exclusive right arm and form a vertical chain, whose upper member Q_t has $y_t < s_0$.

For $k = 3, 4, 5$, the triple $\{Q_1, Q_2, Q_k\}$ has side sum at least H and clears Σ. Hence Q_k cannot lie in the right arm. If Q_k x-overlapped Q_t, it would lie above Q_t, since $y_k \geq s_0 > y_t$, and the squares Q_1, Q_2, Q_k would form a clearing vertical chain. Hence Q_3, Q_4, Q_5 are all x-disjoint from Q_t. They are mutually x-disjoint as members of the forced four-chain. The quadruple $\{t, 3, 4, 5\}$ differs from $\{3, 4, 5, 6\}$ and therefore clears Σ, a contradiction. □

Theorem 5.5. *For every $s \in \ell_2^{\downarrow}$,*

$$L_7^*(s) = F_7(s) = \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}),$$

where $\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}$ are the costs in Proposition 5.1.

Proof. Proposition 5.1 gives $L_7^* \leq F_7 = \min(\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}})$. For the reverse inequality, note that $L_7^* \geq L_6^*$ by Lemma 2.2. If $F_7 = L_6^*$, this suffices. Otherwise $F_7 > L_6^* \geq s_0 + s_1$, and $F_7 \leq R$, $F_7 \leq \max(C_0, C_3)$, and $F_7 \leq \max(W, H)$. If $C_3 \geq F_7$, use (5.4). Otherwise $C_0 \geq F_7$. If $W \geq F_7$, use (5.3). Otherwise $H \geq F_7$. Since $C_0 \geq F_7 > s_0 + s_1$, we have $s_5 + s_6 > s_1$, whence

$$W_{24} - H = s_4 + s_6 - s_1 \geq s_5 + s_6 - s_1 > 0.$$

All three terms in (5.5) are then at least F_7. Thus, $L_7^* \geq F_7$. □

6. Minimality of convex representations

To end the paper, we prove that the complete strategy families of the preceding sections are minimal. Every strategy cost ℓ_σ is a maximum of finitely many linear functions, namely its chain sums, and is therefore convex.

Lemma 6.1 (Counting convex costs)**.** *Let D be a convex cone and $F : D \to \mathbb{R}$ positively homogeneous. Suppose $p_1, \ldots, p_N \in D$ satisfy*

$$F(p_i + p_j) > F(p_i) + F(p_j) \quad \text{for all } i < j. \tag{6.1}$$

Every representation $F = \min_{1 \leq k \leq m} h_k$ with convex h_k on D requires $m \geq N$.

Proof. Each $h_k \geq F$. For each i, choose k_i such that $h_{k_i}(p_i) = F(p_i)$. If $k_i = k_j = k$ for some $i < j$, positive homogeneity of F and convexity of h_k give

$$F(p_i + p_j) = 2F\left(\frac{p_i + p_j}{2}\right) \leq 2h_k\left(\frac{p_i + p_j}{2}\right) \leq F(p_i) + F(p_j),$$

contradicting (6.1). Thus, $k_1, \ldots, k_N$ are distinct, so $m \geq N$. □

Theorem 6.2. *For $n = 6, 7$, the least numbers m of real-valued convex functions $h_1, \ldots, h_m$ on $\ell_2^{\downarrow}$ with $L_n^*(s) = \min_k h_k(s)$ for every $s \in \ell_2^{\downarrow}$ are 1 and 3, respectively. These are also the minimum cardinalities of complete strategy families.*

Proof. By Theorems 3.8 and 5.5, the strategy of Proposition 3.4 and the three strategies of Proposition 5.1 form complete families of the stated sizes, and their costs are convex. This proves the assertion for $n = 6$.

For seven squares take

$$p_1 = (39, 38, 32, 31, 14, 13, 12), \quad p_2 = (12, 9, 8, 7, 6, 5, 4), \quad p_3 = (16, 10, 9, 8, 7, 6, 5).$$

Evaluation of (5.2) gives

s	p_1	p_2	p_3
$L_7^*(s)$	77	21	26

and

$$L_7^*(p_1 + p_2) = 101 > 98, \quad L_7^*(p_1 + p_3) = 104 > 103, \quad L_7^*(p_2 + p_3) = 48 > 47.$$

Apply Lemma 6.1. □

Corollary 6.3 (Unit-container parameter sets)**.** *The set $X_6 = \{s \in \ell_2^{\downarrow} : L_6^*(s) \le 1\}$ is convex, whereas $X_7 = \{s \in \ell_2^{\downarrow} : L_7^*(s) \le 1\}$ is a union of three convex subsets and is not a union of two convex subsets.*

Proof. The set X_6 is a sublevel set of the convex function L_6^*. By Theorem 5.5, X_7 is the union of the sublevel sets of the three convex strategy costs $\ell_{\text{top}}, \ell_{\text{center}}, \ell_{\text{right}}$. For the last assertion, let p_1, p_2, p_3 be the vectors in the proof of Theorem 6.2, and put $\hat{p}_i = p_i / L_7^*(p_i) \in X_7$. For $i \ne j$, the point $(p_i + p_j)/(L_7^*(p_i) + L_7^*(p_j))$ lies on the segment joining $\hat{p}_i$ and $\hat{p}_j$, and its optimal value $L_7^*(p_i + p_j)/(L_7^*(p_i) + L_7^*(p_j))$ is greater than one. Thus, no convex subset of X_7 contains two of $\hat{p}_1, \hat{p}_2, \hat{p}_3$, whereas one of two sets covering X_7 would contain two of them. □

In the notation of [CKK+17], the points $\hat{p}_1, \hat{p}_2, \hat{p}_3$ form a clique in the invisibility graph of X_7, and the convexity number of X_7 is three.

Development of the results and AI assistance

I first read the paper of Moon and Moser in 2016. This project began nearly ten years later, in June 2026, when I started actively looking for a formula for L_6^*, and I obtained a complete proof of the formula for L_7^* in July 2026. Throughout this period I used Claude Opus 4.8 (Anthropic) as a research assistant.

I understand a configuration of squares best by seeing it. In the second half of June 2026, after finding the formula for L_6^*, I used Claude Opus 4.8 to develop a custom interactive tool for drawing and rearranging axis-parallel squares, and this tool was an essential aid in the search for L_7^*. I arranged squares by hand and read candidate terms off the chains that determine the container side. The AI system computed exact optima for integer side vectors with the constraint-programming solver CP-SAT of Google OR-Tools and compared them with the candidate formulas. To find further arrangements, I examined the optimal packings at side vectors where a candidate formula failed. The strategies center and right in Figure 1.2 arose by inserting Q_6 into the six-square arrangement of

Proposition 3.4, and the top strategy was read off optimal packings at side vectors where those insertions failed.

I designed the lower-bound method: assume that a packing has side below the asserted bound, use side sums exceeding that side to exclude chains, and follow the forced overlaps to a contradiction. With the assistance of Claude Opus 4.8, this approach was refined into its present form, built from clearing sets, pins, cornering arguments, and a small number of lower bounds combined by arithmetic. In preparing this manuscript, I used Claude Opus 5.5 (Anthropic) and GPT-6 Astra Pro (OpenAI) to review the mathematical arguments and to improve the exposition; Claude Opus 5.5 also assisted with typesetting, figures, and exact-arithmetic checks of the constructions. I have checked all proofs and take full responsibility for the content of this paper.

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