BEURLING'S CRITERION AND EXTREMAL METRICS FOR FUGLEDE MODULUS

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Abstract. For each $1 \leq p < \infty$, we formulate a necessary and sufficient condition for an admissible metric to be extremal for the Fuglede p-modulus of a system of measures. When $p = 2$, this characterization generalizes Beurling's criterion, a sufficient condition for an admissible metric to be extremal for the extremal length of a planar curve family. In addition, we prove that every Borel function $\varphi : \mathbb{R}^n \to [0, \infty]$ satisfying $0 < \int \varphi^p < \infty$ is extremal for the p-modulus of some curve family in $\mathbb{R}^n$.

1. Introduction

In this note we take a close look at extremal metrics for systems of measures and families of curves. Let us start by recalling Fuglede's definition of modulus [6]. Fix once and for all a measure space $(X, \mathcal{M}, m)$. A collection of measures $\mathbf{E}$ is a *measure system (over $\mathcal{M}$)* if each measure $\mu \in \mathbf{E}$ is defined on the σ-algebra $\mathcal{M}$. A Borel function $\varphi : X \to [0, \infty]$ is called a *metric* and is said to be *admissible* for $\mathbf{E}$ if $\int \varphi \, d\mu \geq 1$ for all $\mu \in \mathbf{E}$. (We do not identify two metrics which agree m-a.e.) For each $0 < p < \infty$, the *p-modulus* of $\mathbf{E}$ is given by

$$\operatorname{mod}_p \mathbf{E} = \inf \left\{ \int \varphi^p \, dm : \varphi \text{ is admissible for } \mathbf{E} \right\}$$

where $\operatorname{mod}_p \mathbf{E} = \infty$ if admissible metrics for $\mathbf{E}$ do not exist.

Example 1. To pick a concrete setting, take $(X, \mathcal{M}, m) = (\mathbb{R}^n, \mathcal{B}_n, m_n)$ where m_n is the Lebesgue measure on the Borel subsets $\mathcal{B}_n$ of $\mathbb{R}^n$. A *(locally rectifiable) curve* γ in $\mathbb{R}^n$ is a concatenation (disjoint union) of countably many images of one-to-one Lipschitz maps $\gamma_i : [a_i, b_i] \to \mathbb{R}^n$. Each image $\gamma_i([a_i, b_i])$ is called a *piece* of γ; curves may have disjoint or overlapping pieces. (For an alternative definition of a curve, see [17].) The *trace* of a curve γ is the set $\bigcup_i \gamma_i([a_i, b_i])$, i.e. the union of the pieces of γ. For every curve γ in $\mathbb{R}^n$ there is a Borel measure $\tilde{\gamma}$ on $\mathbb{R}^n$ such that the line integral

$$\int_\gamma f \, ds = \sum_i \int_{a_i}^{b_i} f(\gamma_i(t)) |\gamma_i'(t)| dt$$

is given by integration against $\tilde{\gamma}$, i.e. $\int_\gamma f \, ds = \int_{\mathbb{R}^n} f \, d\tilde{\gamma}$ for every Borel function f. (By the area formula $\tilde{\gamma} = \sum_i \tilde{\gamma}_i$ where $\tilde{\gamma}_i = H^1 \llcorner \gamma_i([a_i, b_i])$ is the 1-dimensional Hausdorff measure restricted to the set $\gamma_i([a_i, b_i])$, e.g. see [5].) For all $1 \leq p < \infty$, the *p-modulus* of a family of curves Γ in $\mathbb{R}^n$ is defined in terms of Fuglede modulus

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by $\operatorname{mod}_p \Gamma = \operatorname{mod}_p \{\tilde{\gamma} : \gamma \in \Gamma\}$. Therefore, $\operatorname{mod}_p \Gamma = \inf_\varphi \int_{\mathbb{R}^n} \varphi^p \, dm_n$ where the infimum runs over all Borel functions $\varphi \geq 0$ such that $\int_\gamma \varphi \, ds \geq 1$ for all $\gamma \in \Gamma$. (One could similarly define $\operatorname{mod}_p \Gamma$ for $0 < p < 1$, but this quantity is always zero.) In the plane, the *extremal length* $\lambda(\Gamma) = 1/\operatorname{mod}_2 \Gamma$ of a curve family Γ in $\mathbb{R}^2$ is often used instead of its modulus.

An *atom* in a σ-algebra $\mathcal{M}$ is a nonempty set $A \in \mathcal{M}$ with the property $B \subsetneq A$, $B \in \mathcal{M} \Rightarrow B = \emptyset$. That is, the only proper measurable subset of an atom is the empty set. If $\varphi : X \to [0, \infty]$ is a Borel function on $(X, \mathcal{M})$, then φ is constant on each atom of $\mathcal{M}$. Given an atom $A \in \mathcal{M}$, the *atomic measure* δ_A is defined by the rule $\delta_A(S) = 1$ if $A \subset S$ and $\delta_A(S) = 0$ otherwise; $\int \varphi \, d\delta_A = \varphi(A)$ for all φ and A.

Example 2. Let $\mathcal{K} = \{K_1, \dots, K_\ell\}$ be a finite set of pairwise disjoint compact subsets of the Riemann sphere $\hat{\mathbb{C}}$, and let $\Omega \subset \hat{\mathbb{C}}$ be an open set. The *transboundary measure space* $(\hat{\mathbb{C}}, \mathcal{M}_\mathcal{K}, m_{\Omega,\mathcal{K}})$ is defined as follows. Let $\mathcal{B}(\hat{\mathbb{C}} \setminus K)$ denote the Borel σ-algebra on the complement of $K = \bigcup_{i=1}^\ell K_i$. Then $\mathcal{M}_\mathcal{K}$ is the smallest σ-algebra generated by $\mathcal{B}(\hat{\mathbb{C}} \setminus K) \cup \mathcal{K}$. The atoms of $\mathcal{M}_\mathcal{K}$ are the singletons $\{x\}$ with $x \in \hat{\mathbb{C}} \setminus K$ and the sets $K_1, \dots, K_\ell$. We define the measure $m_{\Omega,\mathcal{K}} = H^2 \llcorner (\Omega \setminus K) + \sum_{i=1}^\ell \delta_{K_i}$ where $H^2 \llcorner (\Omega \setminus K)$ is 2-dimensional Hausdorff measure on $\Omega \setminus K$. Let $\gamma : [a, b] \to \hat{\mathbb{C}}$ be a one-to-one continuous map, let $\operatorname{Im} \gamma = \gamma([a, b])$ be its image, and assume that $\operatorname{Im} \gamma \cap (\Omega \setminus K)$ is locally rectifiable. Then we define a measure $\hat{\gamma}$ on $(\hat{\mathbb{C}}, \mathcal{M}_\mathcal{K})$ by

$$\hat{\gamma} = H^1 \llcorner \operatorname{Im} \gamma \cap (\Omega \setminus K) + \sum_{i : \operatorname{Im} \gamma \cap K_i \neq \emptyset} \delta_{K_i}.$$

Suppose that $(X, \mathcal{M}, m) = (\hat{\mathbb{C}}, \mathcal{M}_\mathcal{K}, m_{\Omega,\mathcal{K}})$. The *transboundary modulus* $\operatorname{mod}_{\Omega,\mathcal{K}} \Gamma$ of a collection Γ of one-to-one continuous maps $\gamma : [a, b] \to \hat{\mathbb{C}}$ is defined via Fuglede modulus by $\operatorname{mod}_{\Omega,\mathcal{K}} \Gamma = \operatorname{mod}_2 \{\hat{\gamma} : \gamma \in \Gamma \text{ and } \operatorname{Im} \gamma \cap (\Omega \setminus K) \text{ is locally rectifiable}\}$. Thus, an admissible metric $\varphi : \hat{\mathbb{C}} \to [0, \infty]$ satisfies

$$\int_{\operatorname{Im} \gamma \cap (\Omega \setminus K)} \varphi \, ds + \sum_{i : \operatorname{Im} \gamma \cap K_i \neq \emptyset} \varphi(K_i) \geq 1$$

for every $\gamma \in \Gamma$ such that $\operatorname{Im} \gamma \cap (\Omega \setminus K)$ is locally rectifiable, and

$$\operatorname{mod}_{\Omega,\mathcal{K}} \Gamma = \inf_\varphi \int_{\Omega \setminus K} \varphi^2 \, dH^2 + \sum_{i=1}^\ell \varphi(K_i)^2$$

where the infimum runs over all admissible metrics $\varphi : \hat{\mathbb{C}} \to [0, \infty]$. The reciprocal $\lambda_{\Omega,\mathcal{K}}(\Gamma) = 1/\operatorname{mod}_{\Omega,\mathcal{K}} \Gamma$ of transboundary modulus is *transboundary extremal length*.

The definition of extremal length is due to Beurling and has roots in the classical length-area principle for conformal maps; see Jenkins [12] for a historical overview. Since the introduction of extremal length by Ahlfors and Beurling [2], the modulus of a curve family has become a widely-used tool, employed in geometric function theory [1, 7, 15], quasiconformal and quasiregular mappings [17, 18], dynamical systems [8, 13], and analysis on metric spaces [10, 11]. The transboundary extremal length of a curve family was introduced by Schramm [16] to study uniformization on countably-connected domains. Recently Bonk [4] used transboundary modulus in a crucial way to obtain uniformization results on Sierpiński carpets in the plane. For applications of modulus of measures, see Fuglede's original applications in [6], Hakobyan's work on the conformal dimension of sets [9], and Bishop and Hakobyan's recent paper on the frequency of dimension distortion by quasisymmetric maps [3].

A few nice properties of modulus are apparent from the definition. First if $\mathbf{E} \subset \mathbf{F}$ then $\operatorname{mod}_p \mathbf{E} \leq \operatorname{mod}_p \mathbf{F}$. Second $\operatorname{mod}_p \bigcup_{i=1}^{\infty} \mathbf{E}_i \leq \sum_{i=1}^{\infty} \operatorname{mod}_p \mathbf{E}_i$ for any sequence of measure systems. Since $\operatorname{mod}_p \emptyset = 0$, this says that modulus is an outer measure on measure systems. A third useful property is that every admissible metric gives an upper bound on modulus, i.e. $\operatorname{mod}_p \mathbf{E} \leq \int \varphi^p \, dm$ for all admissible metrics φ.

If the infimum in the definition of the modulus of a measure system $\mathbf{E}$ is obtained by an admissible metric φ, i.e. if $\operatorname{mod}_p \mathbf{E} = \int \varphi^p \, dm$, then the metric φ is said to be *extremal* for the p-modulus of $\mathbf{E}$. Naturally one may ask whether an extremal metric always exists, and if so, to what extent is an extremal metric uniquely determined. Unfortunately simple examples (see Example 5 below) show that the existence and uniqueness of extremal metrics fails for general measure systems. Nevertheless, Fuglede [6] proved that when $1 < p < \infty$, a measure system always admits an extremal metric, after removing an exceptional system of measures.

Fuglede's Lemma. *Let $1 < p < \infty$. Let $\mathbf{E}$ be a measure system. If $\operatorname{mod}_p \mathbf{E} < \infty$, then there exists a measure system $\mathbf{N} \subset \mathbf{E}$ such that $\operatorname{mod}_p \mathbf{N} = 0$ and $\mathbf{E} \setminus \mathbf{N}$ admits an extremal metric φ.*

The uniqueness of an extremal metric for the p-modulus of a measure system also holds when $1 < p < \infty$, up to redefinition of the metric on a set of m-measure zero. This can be seen as follows. Suppose that $\varphi, \psi \in L^p(m)$ are two extremal metrics for the p-modulus of a measure system $\mathbf{E}$. Then the averaged metric $\chi = \frac{1}{2}\varphi + \frac{1}{2}\psi$ is still admissible for $\mathbf{E}$ and $(\operatorname{mod}_p \mathbf{E})^{1/p} \leq \|\chi\|_p \leq \frac{1}{2}\|\varphi\|_p + \frac{1}{2}\|\psi\|_p = (\operatorname{mod}_p \mathbf{E})^{1/p}$. Thus, $\|\frac{1}{2}\varphi + \frac{1}{2}\psi\|_p = \frac{1}{2}\|\varphi\|_p + \frac{1}{2}\|\psi\|_p$. By the condition for equality in Minkowski's inequality and the assumption that $\|\varphi\|_p = \|\psi\|_p < \infty$, one obtains $\varphi = \psi$ m-a.e., as desired.

A fundamental problem working with modulus is to identify an extremal metric for a given measure system or curve family if one exists. Beurling found a general sufficient condition which guarantees that an admissible metric for a curve family in the plane is extremal for its extremal length.

Beurling's Criterion (Ahlfors [1], Theorem 4.4)**.** *Let Γ be a curve family in $\mathbb{R}^2$ and let φ be an admissible metric for Γ such that $0 < \int_{\mathbb{R}^2} \varphi^2 < \infty$. Suppose that there exists a curve family Γ_0 in $\mathbb{R}^2$ such that*

(1) *$\Gamma_0 \subset \Gamma$,*
(2) *$\int_\gamma \varphi \, ds = 1$ for every $\gamma \in \Gamma_0$, and*
(3) *for all $f \in L^2(\mathbb{R}^2)$ taking values in $[-\infty, \infty]$: if $\int_\gamma f \, ds \geq 0$ for all $\gamma \in \Gamma_0$, then $\int_{\mathbb{R}^2} f\varphi \geq 0$.*

Then φ is an extremal metric for the extremal length of Γ, i.e. $\lambda(\Gamma) = \left(\int_{\mathbb{R}^2} \varphi^2\right)^{-1}$.

Let us see Beurling's criterion in action, in a standard example.

Example 3. Let R be a rectangle with side lengths $a \leq b$. Let Γ be the family of all curves in R with connected trace which join opposite edges in R (see Figure 1). We claim that $\varphi = \frac{1}{a}\chi_R$ is an extremal metric for Γ, and thus,

$$\lambda(\Gamma) = \left(\int_R \frac{1}{a^2}\right)^{-1} = a/b.$$

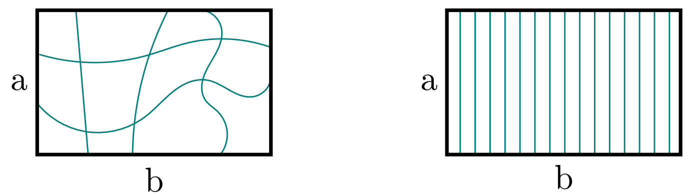

FIGURE 1. Curve families Γ and Γ_0 in Examples 3 and 4

First φ is admissible for Γ, because every curve connecting opposite edges in R travels at least Euclidean distance a (the distance between the edges of length b). Beurling's criterion holds with $\Gamma_0 = \{\gamma(t) : t \in [0,b]\}$ equal to the family of straight line segments connecting (and orthogonal to) a pair of opposite sides of length b. Conditions (1) and (2) hold by definition. And (3) follows from Fubini's theorem: if $\int_{\gamma(t)} f\,ds \geq 0$ for all $\gamma(t) \in \Gamma_0$, then $\int_{\mathbb{R}^2} f\varphi = \frac{1}{a}\int_R f = \frac{1}{a}\int_0^b \int_{\gamma(t)} f\,ds\,dt \geq 0$. Therefore, φ is extremal for $\lambda(\Gamma)$.

The converse to Beurling's criterion fails for the simple reason that Γ may not contain any curves γ such that $\int_\gamma \varphi\,ds = 1$.

Example 4. Once again let R be a rectangle with side lengths $a \leq b$, and let Γ and Γ_0 be the curve families from Example 3. We claim that $\varphi = \frac{1}{a}\chi_R$ is an extremal metric for $\Gamma_* = \Gamma \setminus \Gamma_0$. However, since Γ_* does not contain any curves γ such that $\int_\gamma \varphi\,ds = 1$, we cannot use Beurling's criterion to show that φ is extremal for Γ_*. Let ψ be an admissible metric for Γ_*. Fix $\gamma(t) \in \Gamma_0$. Then one can find a sequence of curves $\gamma^k(t) \in \Gamma_*$ such that $\int_{\gamma^k(t)} \psi\,ds \to \int_{\gamma(t)} \psi\,ds$. (For example, if $\gamma(t) = [0,a]$, then take $\gamma^k(t) = [1/k, 0] \sqcup [0,a]$ where $\sqcup$ denotes concatenation.) In particular, it follows that $\int_{\gamma(t)} \psi\,ds \geq 1$ for all $\gamma(t) \in \Gamma_0$. Integrating across all $\gamma(t) \in \Gamma_0$, invoking Fubini's theorem and applying the Cauchy-Schwarz inequality gives

$$b \leq \int_0^b \int_{\gamma(t)} \psi\,ds\,dt = \int_R \psi \leq \left(\int_R \psi^2\right)^{1/2} (ab)^{1/2} \leq \left(\int_{\mathbb{R}^2} \psi^2\right)^{1/2} (ab)^{1/2}.$$

Thus, $\left(\int_{\mathbb{R}^2} \psi^2\right)^{-1} \leq a/b$ for every metric ψ that is admissible for Γ_*. Since this upper bound is obtained by φ, we conclude that φ is extremal for $\lambda(\Gamma_*)$.

A partial converse to Beurling's criterion is presented in Ohtsuka [14, §2.3] in the special case $\Gamma = \Gamma_0$: *if φ is extremal for Γ, then (3) holds for all $f \in L^2(\mathbb{R}^2)$.* Wolf and Zwiebach [19, p. 38] have also established "a partial local converse to Beurling's criterion" for certain classes of metrics on Riemann surfaces.

2. Statement of Results

The main goal of this note is to show that Beurling's criterion can be modified to become a necessary and sufficient test for extremal metrics. In fact, we establish a characterization of extremal metrics in the general setting of Fuglede modulus, when $1 < p < \infty$ and when $p = 1$.

Theorem 1 (Extremal Metrics in L^p)**.** *Let $1 < p < \infty$. Let $\mathbf{E}$ be a measure system and let φ be an admissible metric for $\mathbf{E}$ such that $\varphi \in L^p(m)$. Then φ is extremal for the p-modulus of $\mathbf{E}$ if and only if*

(B_p) *There exists a measure system* $\mathbf{F}$ *such that*

(a) $\operatorname{mod}_p \mathbf{E} \cup \mathbf{F} = \operatorname{mod}_p \mathbf{E}$,

(b) $\int \varphi \, d\nu = 1$ *for every* $\nu \in \mathbf{F}$, *and*

(c) *for all* $f \in L^p(m)$ *taking values in* $[-\infty, \infty]$: *if* $\int f \, d\nu \geq 0$ *for all* $\nu \in \mathbf{F}$, *then* $\int f \varphi^{p-1} dm \geq 0$.

Theorem 2 (Extremal Metrics in L^1)**.** *Let* $\mathbf{E}$ *be a measure system and let* φ *be an admissible metric for* $\mathbf{E}$ *such that* $\varphi \in L^1(m)$. *Then* φ *is extremal for the* 1*-modulus of* $\mathbf{E}$ *if and only if*

(B_1) *There exists a measure system* $\mathbf{F}$ *such that*

(a) $\operatorname{mod}_1 \mathbf{E} \cup \mathbf{F} = \operatorname{mod}_1 \mathbf{E}$,

(b) $\int \varphi \, d\nu = 1$ *for every* $\nu \in \mathbf{F}$, *and*

(c) *for all* $f \in L^1(m)$ *taking values in* $[-\infty, \infty]$ *such that* $\varphi(x) = 0$ *implies* $f(x) \geq 0$: *if* $\int f \, d\nu \geq 0$ *for all* $\nu \in \mathbf{F}$, *then* $\int f \, dm \geq 0$.

Remark 1. We label the conditions (B_p) in Theorems 1 and 2 in honor of Beurling. When $p = 2$ and $\mathbf{F} \subset \mathbf{E}$, (a) holds vacuously and (B_2) is Beurling's criterion.

Remark 2. In Theorems 1 and 2, if φ is extremal for $\operatorname{mod}_p \mathbf{E}$, then there exists $\mathbf{F}$ satisfying (B_p) such that for every $\nu \in \mathbf{F}$ there exist $\mu \in \mathbf{E}$ and $0 < c \leq 1$ such that $\nu = c\mu$.

Remark 3. In Theorems 1 and 2 the case $\mathbf{F} = \emptyset$ is allowed. Note condition (B_p) holds with $\mathbf{F} = \emptyset$ if and only if $\varphi = 0$ m-almost everywhere.

The proofs of Theorems 1 and 2 will be given in sections 3 and 4, respectively. (A curious reader may jump to the proofs immediately.) We now demonstrate the use of the theorems in a simple, yet enlightening example, which shows the varied behavior of the p-modulus for different values of p.

Example 5. Assume that $A \in \mathcal{M}$ and $0 < m(A) < \infty$. Let $\mathbf{E}_A = \{m \llcorner A\}$ where $m \llcorner A$ denotes the measure m restricted to the set A. Then

$$\operatorname{mod}_p \mathbf{E}_A = \begin{cases} \inf\{m(B)^{1-p} : B \subset A, \ m(B) > 0\}, & \text{if } 0 < p \leq 1, \\ m(A)^{1-p}, & \text{if } 1 \leq p < \infty. \end{cases}$$

This will be checked in four steps.

Let $1 < p < \infty$. We will show that $\varphi_A = m(A)^{-1}\chi_A$ is extremal for $\operatorname{mod}_p \mathbf{E}_A$, and hence, $\operatorname{mod}_p \mathbf{E}_A = \int_A m(A)^{-p} \, dm = m(A)^{1-p}$. Clearly $\varphi_A \in L^p(m)$ and φ_A is admissible for $\mathbf{E}_A$. Let us check that (B_p) holds with $\mathbf{F} = \mathbf{E}_A$. Conditions (a) and (b) hold immediately. For condition (c), $\int f \varphi_A^{p-1} \, dm = m(A)^{1-p} \int_A f \, dm \geq 0$ whenever $\int f \, d(m \llcorner A) \geq 0$. Thus, φ_A is extremal for $\operatorname{mod}_p \mathbf{E}_A$, by Theorem 1.

The case $p = 1$ is similar, except that there is no longer a unique extremal metric. Let $B \subset A$ be any subset such that $m(B) > 0$ and let $\varphi_B = m(B)^{-1}\chi_B$. Then $\varphi_B \in L^1(m)$ and φ_B is admissible for $\mathbf{E}_A$. We will check that (B_1) holds with $\mathbf{F} = \mathbf{E}_A$. Conditions (a) and (b) are immediate. To verify condition (c) of (B_1), assume that $f \in L^1(m)$ takes values in $[-\infty, \infty]$, $f(x) \geq 0$ whenever $\varphi_B(x) = 0$ and $\int f \, d(m \llcorner A) \geq 0$. Then $\int f \, dm = \int_{A^c} f \, dm + \int_A f \, dm \geq 0$, where the first term is non-negative since $\varphi_B(x) = 0$ on A^c. Thus, φ_B is extremal for $\operatorname{mod}_1 \mathbf{E}_A$, by Theorem 2, so that $\operatorname{mod}_1 \mathbf{E}_A = \int \varphi_B \, dm = 1$ for every $B \subset A$ with $m(B) > 0$.

Next let $0 < p < 1$ and suppose that A has subsets of arbitrarily small positive measure. Then we can find a sequence of subsets $B_k \subset A$ with $m(B_k) > 0$ such that $\lim_{k\to\infty} m(B_k) = 0$. The normalized characteristic functions $\varphi_k = m(B_k)^{-1}\chi_{B_k}$

are admissible for $\mathbf{E}_A$. Hence $\operatorname{mod}_p \mathbf{E}_A \le \int \varphi_k^p \, dm = m(B_k)^{1-p} \to 0$ as $k \to \infty$, since $0 < p < 1$. Therefore, $\operatorname{mod}_p \mathbf{E}_A = 0 = \inf\{m(B)^{1-p} : B \subset A,\ m(B) > 0\}$. However, there is no extremal metric for $\operatorname{mod}_p \mathbf{E}_A$, because no function $\psi \ge 0$ satisfies $\int \psi \, d(m \llcorner A) \ge 1$ and $\int \psi^p \, dm = 0$ simultaneously.

Finally, let $0 < p < 1$, but suppose that A does not possess subsets of arbitrarily small positive measure. Then $m \llcorner A = c_1 \delta_{A_1} + \cdots + c_k \delta_{A_k}$ is a linear combination of atomic measures, where each atom $A_i \in \mathcal{M}$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. By relabeling, we may assume that $0 < c_1 \le c_j$ for all $2 \le j \le k$. Let $\rho \ge 0$ be an admissible metric for $\mathbf{E}_A$ such that $\int \rho \, d(m \llcorner A) = 1$. (Here we can ask for equality, because $\mathbf{E}_A$ consists of one element.) Define $\eta_j = \rho(A_j) c_j$ for all j. Then $\sum_{j=1}^k \eta_j = \sum_{j=1}^k (\eta_j / c_j) c_j = \int \rho \, d(m \llcorner A) = 1$. Thus, $0 \le \eta_j \le 1$ for all j, and

$$\int \rho^p \, dm \ge \sum_{j=1}^k (\eta_j / c_j)^p c_j = \sum_{j=1}^k \eta_j^p c_j^{1-p} \ge \sum_{j=1}^k \eta_j c_1^{1-p} = c_1^{1-p}.$$

Since the lower bound $\int \rho^p \, dm \ge c_1^{1-p}$ is obtained by the metric $\rho = m(A_1)^{-1} \chi_{A_1}$, we conclude that $\operatorname{mod}_p \mathbf{E}_A = m(A_1)^{1-p} = \inf\{m(B)^{1-p} : B \subset A,\ m(B) > 0\}$.

Remark 4. With the same notation as in the previous example, $\varphi_A = m(A)^{-1} \chi_A$ also satisfies condition (B_p) with $\mathbf{F} = \mathbf{E}_A$ for $0 < p < 1$. But $\operatorname{mod}_p \mathbf{E}_A \neq \int \varphi_A^p \, dm$ when $0 < p < 1$ unless $m \llcorner A = c\delta_A$. Thus, Example 5 shows that condition (B_p) from Theorem 1 is not a sufficient test for extremal metrics when $0 < p < 1$.

The characterizations of extremal metrics for the p-modulus of measure systems in Theorems 1 and 2 also hold for curve families in $\mathbb{R}^n$. In particular, assuming that an extremal metric for $\mathbf{E} = \{\tilde{\gamma} : \gamma \in \Gamma\}$ exists, one can find a measure system $\mathbf{F}$ satisfying condition (B_p), where $\mathbf{F}$ is also associated to a family of curves in $\mathbb{R}^n$.

Corollary 1 (Extremal Metrics in L^p for Curves)**.** *Let $1 < p < \infty$. Let Γ be a curve family in $\mathbb{R}^n$ and let φ be an admissible metric for Γ such that $\varphi \in L^p(\mathbb{R}^n)$. Then φ is extremal for the p-modulus of Γ if and only if*

- (B'_p) *There exists a curve family Γ' in $\mathbb{R}^n$ such that*
  - (a) $\operatorname{mod}_p \Gamma \cup \Gamma' = \operatorname{mod}_p \Gamma$,
  - (b) $\int_\gamma \varphi \, ds = 1$ *for every* $\gamma \in \Gamma'$, *and*
  - (c) *for all $f \in L^p(\mathbb{R}^n)$ taking values in $[-\infty, \infty]$: if $\int_\gamma f \, ds \ge 0$ for all $\gamma \in \Gamma'$, then $\int_{\mathbb{R}^n} f \varphi^{p-1} \ge 0$.*

Corollary 2 (Extremal Metrics in L^1 for Curves)**.** *Let Γ be a curve family in $\mathbb{R}^n$ and let φ be an admissible metric for Γ such that $\varphi \in L^1(\mathbb{R}^n)$. Then φ is extremal for the* 1*-modulus of Γ if and only if*

- (B'_1) *There exists a curve family Γ' in $\mathbb{R}^n$ such that*
  - (a) $\operatorname{mod}_1 \Gamma \cup \Gamma' = \operatorname{mod}_1 \Gamma$,
  - (b) $\int_\gamma \varphi \, ds = 1$ *for every* $\gamma \in \Gamma'$, *and*
  - (c) *for all $f \in L^1(\mathbb{R}^n)$ taking values in $[-\infty, \infty]$ such that $\varphi(x) = 0$ implies $f(x) \ge 0$: if $\int_\gamma f \, ds \ge 0$ for all $\gamma \in \Gamma'$, then $\int_{\mathbb{R}^n} f \ge 0$.*

Remark 5. In Corollaries 1 and 2, if φ is extremal for $\operatorname{mod}_p \Gamma$, then there exists Γ' satisfying (B'_p) such that every curve $\gamma' \in \Gamma'$ is a *subcurve* of some curve $\gamma \in \Gamma$.

Remark 6. In Corollaries 1 and 2 the case $\Gamma' = \emptyset$ is allowed. Note condition (B'_p) holds with $\Gamma' = \emptyset$ if and only if $\varphi = 0$ Lebesgue almost everywhere.

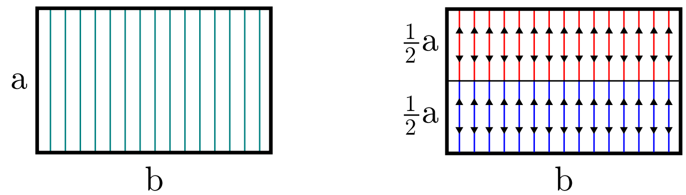

FIGURE 2. Curve families Γ_0 and Γ_1 in Example 6

The auxiliary curve family Γ' that is required to test condition (B_p') is not unique. In the next example, we exhibit disjoint curve families Γ_0 and Γ_1 such that condition (B_2') holds with the auxiliary curve family $\Gamma' = \Gamma_i$, $i = 0, 1$.

Example 6. Let R be a rectangle with side lengths $a \leq b$, and let Γ and Γ_0 be the curve families from Example 3. Above we showed that condition (B_2') (i.e. Beurling's criterion) holds for Γ and $\varphi = \frac{1}{a}\chi_R$ using the auxiliary curve family $\Gamma' = \Gamma_0 \subset \Gamma$. Thus, $\operatorname{mod}_2 \Gamma = \int_R (1/a)^2 = b/a$ by Corollary 1. Alternatively let Γ_1 be the curve family described as follows (see Figure 2). For each $\gamma(t) \in \Gamma_0$, there correspond exactly two curves $\gamma'(t)$ and $\gamma''(t)$ in Γ_1. If the curve $\gamma(t) = [P, Q]$, then the curves $\gamma'(t)$ and $\gamma''(t)$ are given by

$$\gamma'(t) = \left[P, \frac{P+Q}{2}\right] \sqcup \left[\frac{P+Q}{2}, P\right] \quad \text{and} \quad \gamma''(t) = \left[Q, \frac{P+Q}{2}\right] \sqcup \left[\frac{P+Q}{2}, Q\right]$$

where $\sqcup$ denotes concatenation. In other words, each curve in Γ_1 travels along a straight path starting at and perpendicular to an edge of side length b; half-way across to the other side, the curve reverses direction and returns to its starting point. We now check that (B_2') holds for Γ and φ with $\Gamma' = \Gamma_1$. A quick computation shows that $\int_\gamma \varphi\, ds = 1$ for all $\gamma \in \Gamma_1$. Hence condition (b) holds. For (a), we have $\operatorname{mod}_2 \Gamma \leq \operatorname{mod}_2 \Gamma \cup \Gamma_1 \leq \int_{\mathbb{R}^2} \varphi^2 = \operatorname{mod}_2 \Gamma$, since φ is admissible for $\Gamma \cup \Gamma_1$ and since (we already know that) φ is extremal for Γ. It remains to check (c). If $f \in L^2(\mathbb{R}^2)$ and $\int_\gamma f\, ds \geq 0$ for all $\gamma \in \Gamma_1$, then

$$\begin{aligned} 2\int_{\mathbb{R}^2} f\varphi &= \frac{2}{a}\int_R f = \frac{2}{a}\int_0^b \int_{\gamma(t)} f\, ds\, dt = \frac{1}{a}\int_0^b \int_{\gamma(t)\sqcup\gamma(t)} f\, ds\, dt \\ &= \frac{1}{a}\int_0^b \int_{\gamma'(t)\sqcup\gamma''(t)} f\, ds\, dt = \frac{1}{a}\int_0^b \int_{\gamma'(t)} f\, ds\, dt + \frac{1}{a}\int_0^b \int_{\gamma''(t)} f\, ds\, dt \geq 0. \end{aligned}$$

Thus, condition (c) holds too, and we have reached the end of the example.

It is of course possible to specialize Theorems 1 and 2 to other settings. For instance, a reader familiar with analysis on metric spaces will have no difficulty adapting Corollaries 1 and 2 to the metric space setting. In [4, §11], Bonk notes that Beurling's criterion can be adapted to produce a sufficient test for extremal metrics for the transboundary modulus of a curve family in the Riemann sphere. Using Theorem 1 and the proof of Corollary 1, one can also formulate a necessary and sufficient test for extremal metrics for transboundary modulus.

So far we have found a characterization of extremal metrics for the p-modulus of a measure system or curve family when $1 \leq p < \infty$. A related problem is to identify those metrics which are extremal for the p-modulus of some measure system or curve family. The next result gives a solution to this problem for measure systems.

Theorem 3. *If $\varphi : X \to [0,\infty]$ is a metric and $\varphi < \infty$ m-a.e., then φ is extremal for the p-modulus of $\mathbf{E}_\varphi = \{\mu$ defined on $\mathcal{M} : \int \varphi\, d\mu \geq 1\}$ for all $0 < p < \infty$.*

Proof. Let $\mathcal{A}$ be the collection of atoms in $\mathcal{M}$. Note that the scaled atomic measure $\mu_A = \varphi(A)^{-1}\delta_A \in \mathbf{E}_\varphi$ for all $A \in \mathcal{A}$ such that $0 < \varphi(A) < \infty$. Let ψ be an admissible metric for $\mathbf{E}_\varphi$. Then $\psi(A)/\varphi(A) = \int \psi\, d\mu_A \geq 1$ when $0 < \varphi(A) < \infty$. Also, $\psi(A) \geq \varphi(A)$ when $\varphi(A) = 0$. Thus, if $\varphi < \infty$ m-a.e., then $\psi \geq \varphi$ m-a.e., and $\int \psi^p\, dm \geq \int \varphi^p\, dm$ for all $0 < p < \infty$. Therefore, $\operatorname{mod}_p \mathbf{E}_\varphi = \int \varphi^p\, dm$ for all $0 < p < \infty$. □

We can establish a similar result for curve families in $\mathbb{R}^n$. The basic philosophy, suggested by the proof of Theorem 3, is that one needs to approximate the measures δ_x at points where $\varphi(x) > 0$ by sequences of curves. See section 6 for details.

Theorem 4. *If $\varphi : \mathbb{R}^n \to [0,\infty]$ is Borel, then φ is extremal for the p-modulus of $\Gamma_\varphi = \{$curve γ in $\mathbb{R}^n : \int_\gamma \varphi\, ds \geq 1\}$ for all $1 \leq p < \infty$ such that $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$.*

The plan for the remainder of the note is as follows. In the next two sections, we prove the characterizations of extremal metrics for the p-modulus of a measure system from above, in the cases $1 < p < \infty$ (section 3) and $p = 1$ (section 4). Then we turn our attention to extremal metrics for families of curves in $\mathbb{R}^n$. In section 5, we show how the proofs of Theorems 1 and 2 must be modified to obtain Corollaries 1 and 2. Finally, we give the proof of Theorem 4 in section 6.

3. Proof of Theorem 1 (Extremal Metrics in L^p)

Let $1 < p < \infty$. Let $\mathbf{E}$ be a measure system and let φ be an admissible metric for $\mathbf{E}$ such that $\varphi \in L^p(m)$. If $\varphi = 0$ m-a.e., then φ is extremal for the p-modulus of $\mathbf{E}$ and condition (B_p) holds with $\mathbf{F} = \emptyset$. Thus, we assume that $0 < \int \varphi^p\, dm < \infty$.

We shall start with the proof that condition (B_p) implies that φ is extremal, by mimicking the proof of Beurling's criterion in Ahlfors [1]. Suppose that (B_p) holds for some measure system $\mathbf{F}$ satisfying (a), (b) and (c). Since the metric φ is admissible for $\mathbf{E}$, φ is also admissible for $\mathbf{E} \cup \mathbf{F}$, by (b). Let ψ be a competing admissible metric for $\operatorname{mod}_p \mathbf{E} \cup \mathbf{F}$, so that $\int \psi^p\, dm \leq \int \varphi^p\, dm < \infty$. Then $\int \psi\, d\nu \geq 1 = \int \varphi\, d\nu$ for all $\nu \in \mathbf{F}$, by (b). Hence $f = \psi - \varphi \in L^p(m)$ and $\int f\, d\nu \geq 0$ for all $\nu \in \mathbf{F}$. By (c), we conclude that $\int (\psi - \varphi)\varphi^{p-1} \geq 0$. Then

$$\int \varphi^p\, dm \leq \int \psi \varphi^{p-1}\, dm \leq \left(\int \psi^p\, dm\right)^{1/p} \left(\int \varphi^p\, dm\right)^{(p-1)/p} \tag{1}$$

where the second inequality is Hölder's inequality. Since $0 < \int \varphi^p\, dm < \infty$, we get that $\int \varphi^p\, dm \leq \int \psi^p\, dm$. Thus, φ is extremal for the p-modulus of $\mathbf{E} \cup \mathbf{F}$. Finally, $\operatorname{mod}_p \mathbf{E} \leq \int \varphi^p\, dm = \operatorname{mod}_p \mathbf{E} \cup \mathbf{F} = \operatorname{mod}_p \mathbf{E}$, by (a). Therefore, φ is extremal for the p-modulus of $\mathbf{E}$.

For the reverse direction, we require a short lemma.

Lemma 1. *Let $1 < p < \infty$. If $\varphi, f \in L^p(m)$ take values in $[-\infty,\infty]$ and $\varphi \geq 0$, then*

$$\int \left[(\varphi + \varepsilon f)^+\right]^p\, dm = \int_{\{\varphi + \varepsilon f > 0\}} \left[\varphi^p + p\,\varepsilon f \varphi^{p-1}\right]\, dm + o(\varepsilon) \cdot \varepsilon$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. Let $1 < p < \infty$ and let $\varphi, f \in L^p(m)$ be given. Assume that the functions φ and f take values in $[-\infty, \infty]$ and $\varphi \geq 0$. Fix $\varepsilon \neq 0$ and set $P = \{\varphi + \varepsilon f > 0\}$. By the mean value theorem, for all $x \in P$ such that $\varphi(x)$ and $f(x)$ are both finite, there exists $\delta = \delta(x)$ between 0 and ε such that $(\varphi+\varepsilon f)^p - \varphi^p = \left[p(\varphi + \delta f)^{p-1} f\right] \varepsilon$. In particular, this holds at m-a.e. $x \in P$, because $\varphi, f \in L^p(m)$, and the function $\delta : P \to \mathbb{R}$ is measurable, because φ and f are measurable. Hence

$$\int_P (\varphi + \varepsilon f)^p \, dm = \int_P \left[\varphi^p + p\,\varepsilon f \varphi^{p-1}\right] \, dm + \varepsilon \int_P pf \left[(\varphi + \delta f)^{p-1} - \varphi^{p-1}\right] \, dm.$$

The lemma follows, because the second integral in the displayed equation vanishes as $\varepsilon \to 0$ by the dominated convergence theorem. □

Now suppose that φ is extremal for the p-modulus of $\mathbf{E}$. Break $\mathbf{E} = \mathbf{E}_0 \cup \mathbf{E}_\infty$ into a union of two measure systems where $\mathbf{E}_0 = \{\mu \in \mathbf{E} : 1 \leq \int \varphi \, d\mu < \infty\}$ and $\mathbf{E}_\infty = \{\mu \in \mathbf{E} : \int \varphi \, d\mu = \infty\}$. Since $\varphi \in L^p(m)$, we have $\operatorname{mod}_p \mathbf{E}_\infty = 0$, because $\varepsilon\varphi$ is admissible for $\mathbf{E}_\infty$ for all $\varepsilon > 0$. It follows that $\operatorname{mod}_p \mathbf{E}_0 = \operatorname{mod}_p \mathbf{E} = \int \varphi^p \, dm$; that is, φ is extremal for the p-modulus of $\mathbf{E}_0$, as well. Moreover, $\mathbf{E}_0$ is nonempty, since $\operatorname{mod}_p \mathbf{E}_0 > 0$. Recall that we want to show that condition (B_p) holds. Assign $\mathbf{F}$ to be the family of all measures ν defined on $\mathcal{M}$ such that $\int \varphi \, d\nu = 1$. Thus, (b) is satisfied by the definition of $\mathbf{F}$. To verify (a), simply note that

$$\operatorname{mod}_p \mathbf{E} \leq \operatorname{mod}_p \mathbf{E} \cup \mathbf{F} \leq \int \varphi^p \, dm = \operatorname{mod}_p \mathbf{E},$$

since φ is admissible for $\mathbf{E} \cup \mathbf{F}$ and φ is extremal for the p-modulus of $\mathbf{E}$. It remains to establish (c). Assume that $f \in L^p(m)$ takes values in $[-\infty, \infty]$ and $\int f \, d\nu \geq 0$ for every $\nu \in \mathbf{F}$. Then for all $\varepsilon > 0$ the metric $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$ belongs to $L^p(m)$ and $\int \varphi_\varepsilon \, d\nu \geq \int (\varphi + \varepsilon f) \, d\nu \geq \int \varphi \, d\nu = 1$ for every $\nu \in \mathbf{F}$. If $\mu \in \mathbf{E}_0$, then there exists $0 < c \leq 1$ such that $c\mu \in \mathbf{F}$ so that $\int \varphi_\varepsilon \, d\mu \geq c \int \varphi_\varepsilon \, d\mu = \int \varphi_\varepsilon \, d(c\mu) \geq 1$. Hence the metric φ_ε is also admissible for $\mathbf{E}_0$. Thus,

$$\int \varphi^p \, dm = \operatorname{mod}_p \mathbf{E}_0 \leq \int \varphi_\varepsilon^p \, dm = \int \left[(\varphi + \varepsilon f)^+\right]^p \, dm.$$

Then, Lemma 1 gives $\int \varphi^p \, dm \leq \int_{P_\varepsilon} \left[\varphi^p + p\,\varepsilon f \varphi^{p-1}\right] \, dm + o(\varepsilon) \cdot \varepsilon$, where the set $P_\varepsilon = \{\varphi + \varepsilon f > 0\}$ and $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows that

$$p\,\varepsilon \int_{P_\varepsilon} f \varphi^{p-1} \, dm \geq \int_{X \setminus P_\varepsilon} \varphi^p \, dm - o(\varepsilon) \cdot \varepsilon \geq -o(\varepsilon) \cdot \varepsilon.$$

Dividing through by $p\,\varepsilon$ and letting $\varepsilon \to 0+$, we obtain

$$\int f \varphi^{p-1} \, dm = \lim_{\varepsilon \to 0+} \int_{P_\varepsilon} f \varphi^{p-1} \, dm \geq 0,$$

by the dominated convergence theorem. Therefore, condition (B_p) holds if φ is extremal for the p-modulus of $\mathbf{E}$.

4. Proof of Theorem 2 (Extremal Metrics in L^1)

Let $\mathbf{E}$ be a measure system and let φ be an admissible metric for $\mathbf{E}$ such that $\varphi \in L^1(m)$. If $\varphi = 0$ m-a.e., then φ is extremal for the 1-modulus of $\mathbf{E}$ and condition (B_1) holds with $\mathbf{F} = \emptyset$. Thus, we assume that $0 < \int \varphi \, dm < \infty$.

Suppose that condition (B_1) holds. Let ψ be an admissible metric for $\mathbf{E} \cup \mathbf{F}$ with $\psi \in L^1(m)$. Then $\int \psi \, d\nu \geq 1 = \int \varphi \, d\nu$ for every $\nu \in \mathbf{F}$, by (b). Hence the function $f = \psi - \varphi \in L^1(m)$ takes values in $[-\infty, \infty]$, $\int f \, d\nu \geq 0$ for all $\nu \in \mathbf{F}$

and $f(x) \geq 0$ whenever $\varphi(x) = 0$. Since f satisfies the hypothesis of (c), we obtain $\int(\psi - \varphi)\, dm \geq 0$. That is, $\int \varphi\, dm \leq \int \psi\, dm$, for every admissible metric ψ. Thus, φ is extremal for the 1-modulus of $\mathbf{E} \cup \mathbf{F}$. It follows that $\operatorname{mod}_1 \mathbf{E} \leq \int \varphi\, dm = \operatorname{mod}_1 \mathbf{E} \cup \mathbf{F} = \operatorname{mod}_1 \mathbf{E}$, by (a). Therefore, φ is extremal for the 1-modulus of $\mathbf{E}$.

Conversely, suppose that φ is extremal for the 1-modulus of $\mathbf{E}$. Then φ is also extremal for the 1-modulus of $\mathbf{E}_0 = \{\mu \in \mathbf{E} : 1 \leq \int \varphi\, d\mu < \infty\}$. We want to check that condition (B_1) holds. Assign $\mathbf{F}$ to be the family of all measures ν defined on $\mathcal{M}$ such that $\int \varphi\, d\nu = 1$. Then (b) is satisfied automatically. For (a), $\operatorname{mod}_1 \mathbf{E} \leq \operatorname{mod}_1 \mathbf{E} \cup \mathbf{F} \leq \int \varphi\, dm = \operatorname{mod}_1 \mathbf{E}$, since φ is admissible for $\mathbf{E} \cup \mathbf{F}$ and φ is extremal for the 1-modulus of $\mathbf{E}$. It remains to verify (c). Assume that $f \in L^1(m)$ takes values in $[-\infty, \infty]$, $\int f\, d\nu \geq 0$ for all $\nu \in \mathbf{F}$ and $f(x) \geq 0$ whenever $\varphi(x) = 0$. For all $\varepsilon > 0$ the metric $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$ belongs to $L^1(m)$, and moreover, satisfies $\int \varphi_\varepsilon\, d\nu \geq \int (\varphi + \varepsilon f)\, d\nu \geq \int \varphi\, d\nu = 1$ for every $\nu \in \mathbf{F}$. Now, for all $\mu \in \mathbf{E}_0$, there exists $0 < c \leq 1$ such that $c\mu \in \mathbf{F}$. Thus, $\int \varphi_\varepsilon\, d\mu \geq c \int \varphi_\varepsilon\, d\mu = \int \varphi_\varepsilon\, d(c\mu) \geq 1$ for all $\mu \in \mathbf{E}_0$. This shows that the metric φ_ε is also admissible for $\mathbf{E}_0$, and hence,

$$\int \varphi\, dm = \operatorname{mod}_1 \mathbf{E}_0 \leq \int \varphi_\varepsilon\, dm = \int (\varphi + \varepsilon f)^+\, dm = \int_{P_\varepsilon} (\varphi + \varepsilon f)\, dm,$$

where $P_\varepsilon = \{\varphi + \varepsilon f > 0\}$. This yields $\int_{P_\varepsilon} f\, dm \geq \varepsilon^{-1} \int_{X \setminus P_\varepsilon} \varphi\, dm \geq 0$ for all $\varepsilon > 0$. As $\varepsilon \to 0+$, the characteristic functions χ_{P_ε} converge m-a.e. to the function χ_P where $P = \{\varphi > 0\} \cup \{\varphi = 0, f > 0\}$ (convergence at $x \in X$ fails if $f(x) = -\infty$). Therefore, $\int_P f\, dm \geq 0$, and because we assumed that $f(x) \geq 0$ whenever $\varphi(x) = 0$, we obtain $\int f\, dm = \int_P f\, dm + \int_{\{\varphi = 0, f = 0\}} f\, dm \geq 0$, as well. This completes the proof that condition (B_1) holds whenever φ is extremal for the 1-modulus of $\mathbf{E}$.

5. Modification for Curve Families in $\mathbb{R}^n$

The conditions (B_p') of Corollaries 1 and 2 are sufficient tests for metrics to be extremal for $\operatorname{mod}_p \Gamma$ by Theorems 1 and 2. To verify that the conditions (B_p') are also necessary, the proofs of Theorems 1 and 2 can be modified, as follows.

Let $1 \leq p < \infty$. Let $\Gamma \subset \mathbb{R}^n$ be a curve family and let φ be an extremal metric for the p-modulus of Γ such that $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$. Then the metric φ is also extremal for the p-modulus of $\Gamma_0 = \{\gamma \in \Gamma : 1 \leq \int_\gamma \varphi\, ds < \infty\}$. We want to check that condition (B_p') holds. Assign Γ' to be the family of all curves γ in $\mathbb{R}^n$ such that $\int_\gamma \varphi\, ds = 1$. Then (b) holds by definition. To show (a), simply note that $\operatorname{mod}_p \Gamma \leq \operatorname{mod}_p \Gamma \cup \Gamma' \leq \int_{\mathbb{R}^n} \varphi^p = \operatorname{mod}_p \Gamma$, because φ is admissible for $\Gamma \cup \Gamma'$ and φ is extremal for the p-modulus of Γ. It remains to verify (c). Assume that $f \in L^p(\mathbb{R}^n)$ takes values in $[-\infty, \infty]$ and $\int_\gamma f\, ds \geq 0$ for all $\gamma \in \Gamma'$. In the case $p = 1$, also assume that $f(x) \geq 0$ whenever $\varphi(x) = 0$. For all $\varepsilon > 0$, the metric $\varphi_\varepsilon = (\varphi + \varepsilon f)^+ \geq 0$ belongs to $L^p(\mathbb{R}^n)$. Moreover, $\int_\gamma \varphi_\varepsilon\, ds \geq \int_\gamma (\varphi + \varepsilon f)\, ds \geq \int_\gamma \varphi\, ds = 1$ for all $\gamma \in \Gamma'$. If $\gamma \in \Gamma_0$, then

$$1 \leq \int_\gamma \varphi\, ds = \sum_i \int_{a_i}^{b_i} \varphi(\gamma_i(t))|\gamma_i'(t)|dt < \infty.$$

Since each term in the line integral is non-negative and finite, the function

$$c \mapsto \int_{a_i}^{c} \varphi(\gamma_i(t))|\gamma_i'(t)|dt$$

on $[a_i, b_i]$ is continuous for each i. Hence we can pick $c_i \in [a_i, b_i]$ for all i in such a way that the subcurve $\gamma_1 = \bigsqcup_i \gamma([a_i, c_i])$ of γ satisfies $\int_{\gamma_1} \varphi\, ds = 1$. This means that $\gamma_1 \in \Gamma'$. Thus, $\int_\gamma \varphi_\varepsilon\, ds \geq \int_{\gamma_1} \varphi_\varepsilon\, ds \geq 1$. This shows that φ_ε is also admissible for Γ_0. Hence

$$\int_{\mathbb{R}^n} \varphi^p = \mathrm{mod}_p\, \Gamma_0 \leq \int_{\mathbb{R}^n} \varphi_\varepsilon^p = \int_{\mathbb{R}^n} \left[(\varphi + \varepsilon f)^+\right]^p.$$

To finish checking (c), one can now proceed as above. Follow the argument from section 3, when $1 < p < \infty$, and follow the argument from section 4, when $p = 1$.

6. Proof of Theorem 4

Suppose that $\varphi : \mathbb{R}^n \to [0, \infty]$ is a Borel function and let Γ_φ be the family of all curves γ in $\mathbb{R}^n$ such that $\int_\gamma \varphi\, ds \geq 1$. Fix any $1 \leq p < \infty$ such that $0 < \int_{\mathbb{R}^n} \varphi^p < \infty$. We want to show that φ is extremal for the p-modulus of Γ_φ. For each $y \in \mathbb{R}^n$ let $\ell_y = y + \mathbb{R}e_1 \cong \mathbb{R}$ denote the line through y parallel to the direction $e_1 = (1, 0 \ldots, 0)$. By Fubini's theorem, $\varphi \in L^p(\ell_y)$, $y = (0, \bar{y})$ for H^{n-1}-a.e. $\bar{y} \in \mathbb{R}^{n-1}$. In particular, we also have $\varphi \in L^1_{\mathrm{loc}}(\ell_y)$, $y = (0, \bar{y})$ for H^{n-1}-a.e. $\bar{y} \in \mathbb{R}^{n-1}$. Here, as above and as below, H^s denotes s-dimensional Hausdorff measure. Below $|I|$ denotes the diameter of an interval I.

Lemma 2. *Suppose that $\varphi \in L^1_{\mathrm{loc}}(\ell_y)$. Then, for H^1-a.e. $x \in \ell_y$ such that $\varphi(x) > 0$, there exist a sequence of positive integers $n_k = n_k(x) \to \infty$ and a sequence intervals $I_k = I_k(x) \subset \ell_y$ centered at x with $|I_k| \to 0$ such that $\int_{I_k} \varphi\, dt = 1/n_k$ for all k.*

Proof. Define the function $g_x(r) = \int_{-r}^{r} \varphi(x + te_1)\, dt$ for all $x \in \ell_y$ and $r \geq 0$. Then $\lim_{r \to 0+} g_x(r)/2r = \varphi(x)$ for H^1-a.e. $x \in \ell_y$, by the Lebesgue differentiation theorem. Hence for H^1-a.e. $x \in \ell_y$ such that $\varphi(x) > 0$, there exists $r_0 = r_0(x) > 0$ such that $0 < g_x(r) < \infty$ for all $0 < r \leq r_0$. Since $g_x|_{[0,r_0]}$ is continuous and $g_x(0) = 0$, we can find a sequence of integers $n_k = n_k(x)$ and a sequence of radii $r_k = r_k(x) \to 0$ such that $g_x(r_k) = 1/n_k$. Then $I_k = I_k(x) = x + [-r_k, r_k]e_1 \subset \ell_y$ is a sequence of intervals with the desired property. □

Let $E \subset \mathbb{R}^n$ be the set of points $x \in \mathbb{R}^n$ where the conclusion of Lemma 2 holds, i.e. $x \in E$ if and only if there exists a sequence of positive integers $n_k = n_k(x) \to \infty$ and a sequence of intervals $I_k = I_k(x) \subset \ell_x$ centered at x with $|I_k| \to 0$ such that $\int_{I_k} \varphi\, dt = 1/n_k$. By Fubini's theorem and Lemma 2, we have $x \in E$ for a.e. $x \in \mathbb{R}^n$ such that $\varphi(x) > 0$. We define a curve family $\Gamma' \subset \Gamma_\varphi$ as follows. Choose one pair of sequences $(n_k(x))_{k=1}^\infty$ and $(I_k(x))_{k=1}^\infty$ for each $x \in E$. Then, for each $x \in E$ and $k \geq 1$, define a curve $\gamma_k(x) = \bigsqcup_{i=1}^{n_k} I_k(x)$, i.e. let $\gamma_k(x)$ be a curve which covers the interval $I_k(x)$ exactly $n_k(x)$-times. Set $\Gamma' = \{\gamma_k(x) : x \in E \text{ and } k \geq 1\}$.

To prove that φ is extremal for the p-modulus of Γ_φ, it is enough by either Corollary 1 or Corollary 2 (according to whether $1 < p < \infty$ or $p = 1$) to show that (B'_p) holds for Γ'. To start note $\int_{\gamma_k(x)} \varphi\, ds = n_k(x) \int_{I_k(x)} \varphi\, dt = n_k(x)/n_k(x) = 1$ for all $\gamma_k(x) \in \Gamma'$. This shows that (b) holds. And, since $\Gamma' \subset \Gamma_\varphi$, (a) is true too. To verify (c), assume that $f \in L^p(\mathbb{R}^n)$ takes values in $[-\infty, \infty]$ and $\int_{\gamma_k(x)} f\, ds \geq 0$ for all $\gamma_k(x) \in \Gamma'$. Moreover, if $p = 1$, assume that $f(x) \geq 0$ whenever $\varphi(x) = 0$. By Fubini's theorem and the Lebesgue differentiation theorem,

$$f(x) = \lim_{k \to \infty} \frac{1}{|I_k(x)|} \int_{I_k(x)} f\, dt$$

for a.e. $x \in E$, and in particular, for a.e. $x \in \mathbb{R}^n$ such that $\varphi(x) > 0$. By assumption,

$$\int_{I_k(x)} f\,dt = \frac{1}{n_k(x)} \int_{\gamma_k(x)} f\,ds \geq 0 \quad \text{for all } x \in E \text{ and } k \geq 1.$$

Thus, combining the two displayed equations, $f(x) \geq 0$ at a.e. $x \in \mathbb{R}^n$ such that $\varphi(x) > 0$. It follows that $\int_{\mathbb{R}^n} f\varphi^{p-1} \geq 0$, if $1 < p < \infty$, and $\int_{\mathbb{R}^n} f \geq 0$, if $p = 1$. Hence (c) holds. Therefore, (B_p') holds and φ is extremal for the p-modulus of Γ_φ.

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