SLOWLY VANISHING MEAN OSCILLATIONS: NON-UNIQUENESS OF BLOW-UPS IN A TWO-PHASE FREE BOUNDARY PROBLEM

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Dedicado a Carlos Kenig, un gran maestro y amigo en conmemoración de sus 70 años.

Abstract. In Kenig and Toro's two-phase free boundary problem, one studies how the regularity of the Radon-Nikodym derivative $h = d\omega^-/d\omega^+$ of harmonic measures on complementary NTA domains controls the geometry of their common boundary. It is now known that $\log h \in C^{0,\alpha}(\partial\Omega)$ implies that pointwise the boundary has a unique blow-up, which is the zero set of a homogeneous harmonic polynomial. In this note, we give examples of domains with $\log h \in C(\partial\Omega)$ whose boundaries have points with non-unique blow-ups. Philosophically the examples arise from oscillating or rotating a blow-up limit by an infinite amount, but very slowly.

1. Introduction

In this note, we answer a question about uniqueness of blow-ups in non-variational two-phase free boundary problems for harmonic measure *in the negative.* Throughout, we let $\Omega^+ = \Omega \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ denote complementary unbounded domains with a common boundary $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$. Furthermore, we require that $\Omega^\pm$ belong to the class of NTA domains in the sense of Jerison and Kenig [JK82]. Let $\omega^\pm$ denote harmonic measures on $\Omega^\pm$ with finite poles $X^\pm$ or with poles at infinity (see Kenig and Toro [KT99]). Finally, we assume $\omega^+ \ll \omega^- \ll \omega^+$ and let

$$h = \frac{d\omega^-}{d\omega^+} \tag{1.1}$$

denote the Radon-Nikodym derivative of harmonic measure on one side of the boundary with respect to harmonic measure on the other side. We are interested in understanding how different regularity assumptions on h controls the geometry of $\partial\Omega$.

Following Kenig and Toro [KT06] and Badger [Bad11], we know if $\log h \in \mathrm{VMO}(d\omega^+)$ (vanishing mean oscillation) or $\log h \in C(\partial\Omega)$ (continuous), then the boundary admits a finite decomposition into pairwise disjoint sets,

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_{d_0}, \tag{1.2}$$

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where geometric blow-ups (tangent sets) of $\partial\Omega$ centered at any $Q \in \Gamma_d$ $(1 \leq d \leq d_0)$ are zero sets Σ_p of homogeneous harmonic polynomials (hhp) $p : \mathbb{R}^n \to \mathbb{R}$ of degree d. That is to say, given any boundary point $Q \in \Gamma_d$ and any sequence of scales $r_i > 0$ with $\lim_{i\to\infty} r_i = 0$, there exists a subsequence r_{i_j} and a hhp p of degree d such that

$$\lim_{j\to\infty} \max\left\{\operatorname{excess}\left(\frac{\partial\Omega - Q}{r_{i_j}} \cap B, \Sigma_p\right), \operatorname{excess}\left(\Sigma_p \cap B, \frac{\partial\Omega - Q}{r_{i_j}}\right)\right\} = 0 \tag{1.3}$$

for every ball B in $\mathbb{R}^n$. Here $\operatorname{excess}(S,T) = \sup_{s\in S}\inf_{t\in T}|s-t|$ when $S,T \subset \mathbb{R}^n$ are nonempty and $\operatorname{excess}(\emptyset, T) = 0$; see [BL15] for more information about this mode of convergence of closed sets (the Attouch-Wets topology). Following [Bad13] and [BET17], we further know that the regular set Γ_1 is relatively open, Reifenberg flat with vanishing constant, and has Hausdorff and Minkowski dimensions $n-1$, whereas the singular set $\partial\Omega \setminus \Gamma_1$ is closed and has Hausdorff and Minkowski dimension at most $n-3$.

We remark that the maximum degree d_0 witnessed in the decomposition (1.2) can be bounded in terms of the ambient dimension and the NTA constants of $\Omega^\pm$. When $n = 2$, it is always the case that $\partial\Omega = \Gamma_1$. When $n = 3$, we have $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \cdots \cup \Gamma_{2d_1+1}$ (odd degrees only) and for every odd $d \geq 1$, there exist two-sided domains with $\Gamma_d \neq \emptyset$. In dimensions $n \geq 4$, for every integer $d \geq 1$, even or odd, there exist two-sided domains with $\Gamma_d \neq \emptyset$. See [BET17] for details and [AMT20, PT20, TT22] for additional results on the regularity of Γ_1.

One may ask: Are the blow-ups at each point in $\partial\Omega$ unique? In other words, is the zero set Σ_p in (1.3) independent of choice of the sequence of scales r_i? Under a stronger free boundary regularity hypothesis, the answer is *affirmative*. Following Engelstein [Eng16] and [BET20], we know that if $\log h \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha > 0$ (Hölder continuous), then blow-ups are unique. Moreover, when $\log h \in C^{0,\alpha}(\partial\Omega)$, the regular set Γ_1 is actually a $C^{1,\alpha}$ embedded submanifold and the singular set $\partial\Omega \setminus \Gamma_1$ is $(n-3)$-rectifiable in the sense of geometric measure theory (see e.g. [Mat95]). Below, we supply examples demonstrating that under the weaker regularity hypothesis $\log h \in C(\partial\Omega)$, there may exist points in the boundary that have non-unique blow-ups.

Theorem 1.1. *For each $d \in \{1,3\}$, there exist complementary NTA domains $\Omega^\pm \subset \mathbb{R}^3$ such that $\log h \in C(\partial\Omega)$, but there exists a point in Γ_d at which geometric blow-ups of $\partial\Omega$ are not unique.*

Remark 1.2. In fact, the domains that we construct below have *locally finite perimeter* and *Ahlfors regular* boundaries: that is, there exists $C > 0$ (depending on Ω) such that

$$C^{-1}r^{n-1} \leq \mathcal{H}^{n-1}(\partial\Omega \cap B(Q,r)) \leq Cr^{n-1} \quad \text{for all } Q \in \partial\Omega \text{ and } r > 0, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^n$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Even more, the boundaries of the domains are smooth surfaces outside of a single point.

The basic strategy is to start with a blow-up domain $\Omega_p^\pm = \{X \in \mathbb{R}^n : \pm p(X) > 0\}$ associated to a hhp p of degree d, which has $\log h \equiv 0$ and $0 \in \Gamma_d$. We then deform the domain near the origin by introducing rotations/oscillations at each scale $0 < r \leq 1/100$

so that the magnitude of the oscillation at scale r vanishes as $r \to 0$. The tension in the proof becomes choosing the correct speed of vanishing. On the one hand, by choosing the speed to be sufficiently *quick*, we can guarantee by making estimates on elliptic measure that the deformed domain has $\log h \in C(\partial\Omega)$. On the other hand, by choosing the speed to be sufficiently *slow*, we can guarantee that the deformed domain has uncountably many blow-ups at the origin, each of which are rotations of the original domain.

Remark 1.3. By a suitable modification, the technique introduced in the case $d = 3$ can be used to show existence of domains with $\log h \in C(\partial\Omega)$ and non-unique blow-ups at an isolated point $Q \in \Gamma_d$ for any value of $d \geq 2$. When $d \geq 3$ is odd, the examples can be produced in $\mathbb{R}^3$. When $d \geq 2$ is even, the examples can be produced in $\mathbb{R}^4$.

In a related context, Allen and Kriventsov [AK20] use conformal maps to construct domains $\Omega^\pm = \{u^\pm > 0\} \subset \mathbb{R}^n$ ($n \geq 2$) associated to non-negative subharmonic functions $u^\pm$ for which the Alt-Caffarelli-Friedman functional

$$\Phi(r, u^+, u^-) = \frac{1}{r^4} \int_{B_r(0)} \frac{|\nabla u^+|^2}{|X|^{n-2}} \int_{B_r(0)} \frac{|\nabla u^-|^2}{|X|^{n-2}} \tag{1.5}$$

has a positive limit as $r \to 0$, but whose interface $\partial\Omega = \partial\Omega^+ = \partial\Omega^-$ does not have a unique tangent plane at the origin. It would be interesting to know whether a suitable modification of their examples satisfy $\log h \in C(\partial\Omega)$. For more on the connection between the ACF functional and two-phase free boundary problems for harmonic measure (originally observed by Kenig, Preiss, and Toro [KPT09]), see [AKN22, §2.2] and the references within.

We handle the case $d = 3$ of Theorem 1.1 in §2 and the case $d = 1$ in §3.

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2. The First Example: Non-Unique Singular Tangents

2.1. **Description and Geometric Properties.** We begin with Szulkin's example [Szu79] of a degree 3 hhp,

$$s(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z, \tag{2.1}$$

with the interesting feature that its zero set Σ_s is homeomorphic to $\mathbb{R}^2$. See Figure 2.1. Because Σ_s is a cone (s is homogeneous) and $\Sigma_s \cap S^2$ is a smooth curve[1], it follows that $\Omega_s^\pm = \{(x, y, z) \in \mathbb{R}^3 : \pm s(x, y, z) > 0\}$ are complementary NTA domains. Note that the positive z-axis belongs to Ω_s^+ and the negative z-axis belongs to Ω_s^-, since $s(0, 0, \pm 1) = \pm 1$.

[1]One can check that $\nabla s(x, y, z) = 0 \Leftrightarrow (x, y, z) = (0, 0, 0)$.

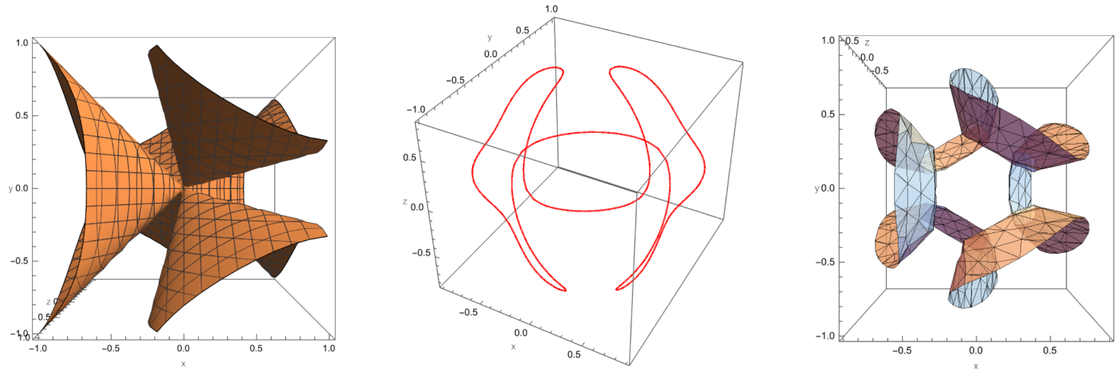

FIGURE 2.1. Left: Szulkin Σ_s, viewed from the z-axis. Center: the curve formed by intersection of Szulkin Σ_s and $\mathbb{S}^2$, viewed from a different angle. Right: Szulkin Σ_s inside of the annulus $1/2 < r < 1$, viewed from the z-axis.

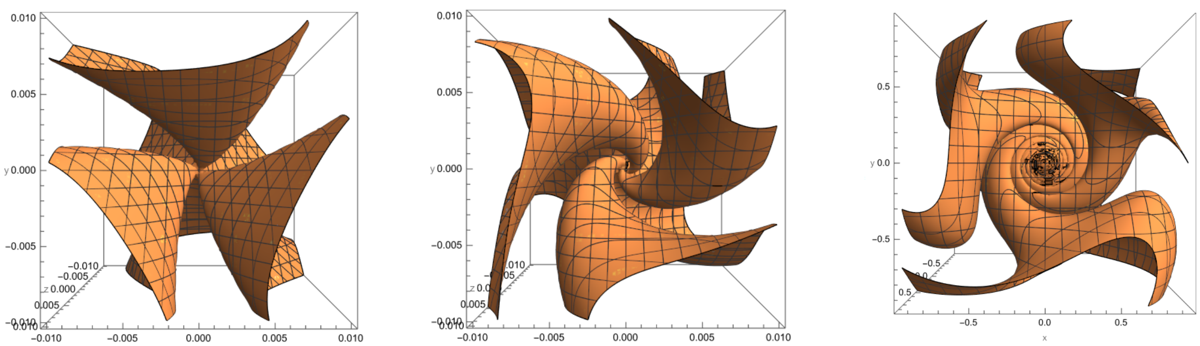

FIGURE 2.2. Examples of twisted Szulkin domains $\Omega^\pm$ defined using various rotation functions $\theta(r)$.
Left: $\theta(r) = \log(-\log(r))$; the domains $\Omega^\pm$ are NTA and $\log h \in C(\partial\Omega)$.
Center: $\theta(r) = -\log(r)$; the domains $\Omega^\pm$ are NTA, but $\log h \notin \mathrm{VMO}(d\omega^+)$.
Right: $\theta(r) = (-\log(r))^2$; the domains $\Omega^\pm$ are not NTA.

To build $\Omega^\pm$, we deform $\Omega_s^\pm$ by rotating spherical shells $\Sigma_s \cap \partial B_r(0)$ in the xy-plane. More precisely, we put $\Omega^\pm = \{\pm s_{\mathrm{twist}} > 0\}$, where $s_{\mathrm{twist}} \equiv s \circ \Phi_{-\theta}$ and $\Phi_{\pm\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ are homeomorphisms given by

$$\Phi_{\pm\theta}(x, y, z) = (x\cos(\pm\theta) - y\sin(\pm\theta), x\sin(\pm\theta) + y\cos(\pm\theta), z), \tag{2.2}$$

$$\theta \equiv \theta(r) := \log(-\log(r)) \quad \text{for all } 0 < r := \sqrt{x^2 + y^2 + z^2} \leq 1/100 \tag{2.3}$$

and we smoothly interpolate to $\theta(r) := 0$ for all $r \geq 1$. See Figure 2.2.

If $s_{\mathrm{twist}}(x, y, z) = 0$, then $\Phi_{-\theta}(x, y, z) \in \Sigma_s$. Hence the interface $\Sigma = \partial\Omega^\pm = \Phi_\theta(\Sigma_s)$. Similarly, $\Omega^\pm = \Phi_\theta(\Omega_s^\pm)$.

Remark 2.1. Let us collect some simple, but useful observations about θ and Φ_θ.

(i) For any $\theta_0 \in [0, 2\pi)$, there exists a sequence $r_i \downarrow 0$ such that $\theta(r_i) = \theta_0 \pmod{2\pi}$, i.e. such that $\min_{k\in\mathbb{Z}} |\theta(r_i) - \theta_0 - 2\pi k| = 0$ for all $i \geq 1$.

(ii) For any sequence $r_i \downarrow 0$, there exists $\theta_0 \in [0, 2\pi)$ and a $r_{i_j} \downarrow 0$ such that $\theta(r_{i_j}) \to \theta_0 \pmod{2\pi}$, i.e. $\lim_{j\to\infty} \min_{k\in\mathbb{Z}} \left|\theta(r_{i_j}) - \theta_0 - 2\pi k\right| = 0$.

(iii) For all $0 < r \leq 1/100$, we have $|\nabla\theta| = 1/(-r\log(r))$ and $|\partial_{ij}\theta| \leq C/(-r^2\log(r))$ for all $1 \leq i, j \leq 3$.

(iv) For all (x, y, z) with $0 < r \leq 1/100$, we can write $D\Phi_\theta = R_\theta + E_\theta$, where

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a rotation matrix and the "error matrix" E_θ is such that $\|E_\theta\|_\infty \leq C/(-\log(r))$, where the norm is the sup norm on the entries of E_θ.

(v) The map $\Phi_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ is a quasiconformal homeomorphism, with $\Phi_\theta^{-1} = \Phi_{-\theta}$. Moreover, Φ_θ is asymptotically conformal at the origin.

Proof. The first property holds since $\theta(r)$ is continuous in r and $\theta(r) \to \infty$ as $r \downarrow 0$. The second property is true by compactness of the torus $\mathbb{R}/2\pi$. The third property is a straightforward computation. By another straightforward (if tedious) computation, $D\Phi_\theta = R_\theta + E_\theta$, where R_θ is as above and E_θ is the rank 1 matrix given by

$$E_\theta = \begin{pmatrix} -x\sin(\theta) - y\cos(\theta) \\ x\cos(\theta) - y\sin(\theta) \\ 0 \end{pmatrix} \begin{pmatrix} \theta_x & \theta_y & \theta_z \end{pmatrix}.$$

Let's examine the (1,1) entry of E_θ. Since $\theta_x = \theta'(r)r_x = \theta'(r)x/r$ and $|x| \leq r$, we have

$$|x\theta_x \sin(-\theta) + y\theta_x\cos(-\theta)| \leq 2r|\theta'(r)| \leq 2/(-\log r).$$

The other non-zero entries of E_θ obey the same estimate. This gives the fourth property. To prove that Φ_θ is quasiconformal (see e.g. [Hei06]), it suffices to check that $\Phi_\theta \in W^{1,n}_{\mathrm{loc}}$ and there exists $1 \leq L < \infty$ such that the a.e. defined singular values $\lambda_1 \leq \lambda_2 \leq \lambda_3$ of $D\Phi_\theta$ satisfy $\lambda_3 \leq L\lambda_1$ a.e. These facts follow from property (iv) and the variational characterization of the minimum and maximum singular values. Furthermore, as $r \downarrow 0$, the maximum ratio of λ_3/λ_1 in B_r goes to 1. Therefore, Φ_θ is asymptotically conformal at the origin. □

The *Hausdorff distance* $\mathrm{HD}(A, B) = \max\{\mathrm{excess}(A, B), \mathrm{excess}(B, A)\}$ for all nonempty sets $A, B \subset \mathbb{R}^n$. Note that $\mathrm{HD}(\lambda A, \lambda B) = \lambda\,\mathrm{HD}(A, B)$ for any dilation factor $\lambda > 0$.

Lemma 2.2 (twisted Szulkin vs. rotations of Szulkin)**.** *If* $r, \epsilon, R > 0$ *and* $0 < Rr \leq 1/100$, *then* $\mathrm{HD}(\Sigma \cap B_{Rr}, R_{\theta(r)}\Sigma_s \cap B_{Rr}) \leq C\max\big(\epsilon r, \sup\{q|\theta(q) - \theta(r)| : \epsilon r \leq q \leq Rr\}\big)$.

Proof. For any $p \in B_{\epsilon r}$, we have $\mathrm{dist}(p, R_{\theta(r)}\Sigma_s \cap B_{Rr}) \leq 2\epsilon r$ and $\mathrm{dist}(p, \Sigma \cap B_{Rr}) \leq 2\epsilon r$, since $0 \in R_{\theta(r)}\Sigma_s$ and $0 \in \Sigma$. Thus, the main issue is to estimate distances inside $B_{Rr} \setminus B_{\epsilon r}$.

Let $p \in \Sigma \cap B_{Rr} \setminus B_{\epsilon r}$, say $p \in \Sigma \cap \partial B_q$ with $\epsilon r \leq q \leq Rr$. Then we may write $p = R_{\theta(q)}x$ for some $x \in \Sigma_s$. Let's estimate $\mathrm{dist}(p, R_{\theta(r)}\Sigma_s \cap B_{Rr})$ from above by the distance of p to the point $y = R_{\theta(r)}x \in R_{\theta(r)}\Sigma_s \cap \partial B_q$. Note that $y = R_{\theta(r)}x = R_{\theta(r)}R_{-\theta(q)}p = R_{\theta(r)-\theta(q)}p$ and $|y| = |p| = q$. Hence

$$\begin{aligned}|p - y| &\leq q|(1,0,0) - (\cos(\theta(q) - \theta(r)), \sin(\theta(q) - \theta(r)), 0)|\\ &= q(2 - 2\cos(\theta(q) - \theta(r)))^{1/2}\\ &\leq Cq|\theta(q) - \theta(r)|,\end{aligned}$$

where the first inequality holds by geometric considerations and the last inequality used the Taylor series expansion for cosine.

A similar inequality holds starting from any $p \in R_{\theta(r)}\Sigma_s \cap B_{Rr} \setminus B_{\epsilon r}$. □

Lemma 2.3. *With $\theta(r) = \log(-\log(r))$, the twisted Szulkin domains $\Omega^\pm$ as defined above are chord-arc domains (i.e. NTA domains with Ahlfors regular boundaries). The interface $\Sigma = \partial\Omega^\pm$ has a continuum of blow-ups at the origin, each of which is a rotation of Σ_s in the xy-plane.*

Proof. The domains $\Omega^\pm = \Phi_\theta(\Omega_s^\pm)$ are NTA, because global quasiconformal maps send NTA domains to NTA domains. Every boundary of an NTA domain is lower Ahlfors regular (see e.g. [Bad12, Lemma 2.3]). Thus, Σ is lower Ahlfors regular. To check upper Ahlfors regularity, first note that Σ_s is upper Ahlfors regular, since Σ_s can be covered by a finite number of Lipschitz graphs. Since $\|\det(D\Phi_\theta)\|_\infty < \infty$, it follows that $\Sigma = \Phi_\theta(\Sigma_s)$ is upper Ahlfors regular, as well.

Let's address the blow-ups of $\partial\Omega$ at the origin. Let $r_i \downarrow 0$ and suppose initially that $\theta(r_i) = \theta_0 \pmod{2\pi}$ for all i. Let $\epsilon(r)$ be a function of r to be specified below. Let $R \gg 1$ be a large radius. By Lemma 2.2, the homogeneity of the Hausdorff distance, and the mean value theorem, we have

$$\begin{aligned}&\mathrm{HD}(r_i^{-1}\Sigma \cap B_R, R_{\theta_0}\Sigma_s \cap B_R)\\ &\qquad\leq Cr_i^{-1}\max\big(\epsilon(r_i)r_i, \sup\{q|\theta(q) - \theta(r_i)| : \epsilon(r_i)r_i \leq q \leq Rr_i\}\big)\\ &\qquad\leq C\max\big(\epsilon(r_i), \sup\{t|\theta(tr_i) - \theta(r_i)| : \epsilon(r_i) \leq t \leq R\}\big)\\ &\qquad\leq C\max\big(\epsilon(r_i), R(R-1)r_i\sup\{|\theta'(tr_i)| : \epsilon(r_i) \leq t \leq R\}\big).\end{aligned}$$

Our task is to choose $\epsilon(r_i)$ so that

$$\lim_{i\to\infty}\epsilon(r_i) = 0 \quad\text{and}\quad \lim_{i\to\infty}\sup\{r_i|\theta'(tr_i)| : \epsilon(r_i) \leq t \leq R\} = 0. \tag{2.4}$$

Since $|\theta'(r)| = 1/(-r\log r)$, we have $\sup\{r_i|\theta'(tr_i)| : \epsilon(r_i) \leq t \leq R\} \leq 1/(-\epsilon(r_i)\log(Rr_i))$ for all sufficiently large i (i.e. for all sufficiently small r_i). Thus, (2.4) is satisfied (e.g.) by choosing $\epsilon(r) = |\log(r)|^{-1/2}$. It follows that $\lim_{i\to\infty}\mathrm{HD}(r_i^{-1}\Sigma \cap B_R, R_{\theta_0}\Sigma_s \cap B_R) = 0$ for all $R > 0$. This implies that Σ/r_i converge to $R_{\theta_0}\Sigma_s$ in the sense of (1.3).

In the general case, starting from any sequence $r_i \downarrow 0$, pass to a subsequence such that $\theta(r_i) \to \theta_0 \pmod{2\pi}$. One can readily check that $R_{\theta(r_i)}\Sigma_s$ converges to $R_{\theta_0}\Sigma_s$ in the Attouch-Wets topology. Therefore, Σ/r_i converges to $R_{\theta_0}\Sigma_s$ in the sense of (1.3) by the special case and the triangle inequality for excess. □

Remark 2.4. For all exponents $0 < p < 1$, the twisted Szulkin domains defined using the rotation function $\theta(r) = (-\log(r))^p$ also satisfy the conclusions of Lemma 2.3. However, there is phase transition at $p = 1$. When $\theta(r) = -\log(r)$, one can show that the blow-ups of Σ are no longer zero sets of hhp. The essential difference is that the "speed of rotation" vanishes as one zooms-in at the origin when $p < 1$, but the "speed of rotation" is constant when $p = 1$. When $p > 1$, the "speed of rotation" goes to infinity as one zooms-in at the origin and the associated twisted Szulkin domains $\Omega^{\pm}$ are not even NTA. See Figure 2.2.

2.2. **Potential Theory for the First Example.** Let $r_i \downarrow 0$ be an arbitrary sequence of radii going to zero and let $K \gg 1$. Recall that $\Sigma \cap (B_{Kr_i} \backslash B_{r_i/K}) = \Phi_\theta(\Sigma_s \cap (B_{Kr_i} \backslash B_{r_i/K}))$. Set

$$\tilde{u}_i^{\pm}(x) = \frac{u^{\pm} \circ \Phi_{-\theta}^{-1}(r_i x) r_i}{\omega^{\pm}(B_{r_i})}, \tag{2.5}$$

where $u^{\pm}$ are the Green's functions with poles at infinity for $\Omega^{\pm}$. Then in $\Omega_s^{\pm} \cap B_K \backslash B_{1/K}$, we have that $\tilde{u}_i^{\pm}$ satisfies

$$-\text{div}(B(r_i x)\nabla -) = 0, \qquad B = (\det D\Phi_\theta)^{-1}(D\Phi_\theta)(D\Phi_\theta)^T$$

and Φ_θ is as in (2.2).

To see that $B(r_i x)$ is Lipschitz regular, we note that Remark 2.1(iii) implies that $\|DB\| \leq \frac{C}{r \log(r)}$. Therefore, using the fundamental theorem of calculus along curves which stay in the annulus $B_K \backslash B_{1/K}$

(2.6)
$$\|B(r_i x) - B(r_i y)\| \leq C r_i |x - y| \sup_{B_{Kr_i} \backslash B_{r_i/K}} \|DB\| \leq \frac{CK}{|\log(r_i)|}|x - y|, \forall x, y \in B_K \backslash B_{1/K},$$

where $C > 0$ is independent of i, K. This uniform Lipschitz continuity immediately implies the next result:

Lemma 2.5. *Let $\alpha \in (0,1), K > 1$. The sequence $\tilde{u}_i^{\pm}$ is pre-compact in $C^{1,\alpha}(\Omega_s^{\pm} \cap B_K \setminus B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^{\pm} \to \kappa s$, uniformly on compacta, where s is the Szulkin polynomial, for some $\kappa > 0$.*

Proof. We see that $\tilde{u}_i^{\pm}$ solves an elliptic PDE with coefficients that are Lipschitz continuous and elliptic with coefficients independent of i. Furthermore,

$$\sup_{B_{4K}} |\tilde{u}_i^{\pm}| \leq C \Leftrightarrow \sup_{B_{4Kr_i}} |u^+| \leq C\frac{\omega^+(B_{r_i})}{r_i}.$$

The latter inequality holds (with a $C > 0$ that depends on K) by the Caffarelli-Fabes-Mortola-Salsa and doubling estimates on harmonic measure in NTA domains, see e.g. [JK82]. Then Schauder theory tells us that $\tilde{u}_i^{\pm}$ are uniformly in $C^{1,\alpha}(\overline{\Omega_s^+} \cap B_K \backslash B_{1/K})$ for any $\alpha \in (0,1)$; see [GT01, Theorem 8.3]. The precompactness follows.

Passing to a subsequence, we get that the sequences converges to functions $\tilde{u}_\infty^{\pm}$, which solves $-\text{div}(B_\infty \nabla \tilde{u}_\infty^{\pm}) = 0$ in $\Omega_s^{\pm} \cap B_K \backslash B_{1/K}$. From (2.6) we see that $B_\infty = \text{Id}$ and so, invoking a diagonal argument, $\tilde{u}_i^\pm \to \tilde{u}_\infty^\pm$, uniformly on compacta in $\mathbb{R}^3$. Furthermore, $\tilde{u}_\infty^\pm$ are positive harmonic functions in $\Omega_s^\pm$ that vanish on $(\Omega_s^\pm)^c$.

Since $(\Omega_s^\pm)^c$ are (global) NTA domains, the boundary Harnack inequality implies that there are scalars $\kappa_\pm > 0$ such that $\tilde{u}_\infty^\pm = \kappa_\pm s$ (see [KT99, Lemma 3.7 and Corollary 3.2]).

To wrap up, let us again note that the points $(0,0,\pm 1) \in \Omega_s^\pm$ are invariant under Φ_θ. Furthermore by symmetry $u^+(0,0,1) = u^-(0,0,-1)$ and $\omega^+(B_r) = \omega^-(B_r)$ for all r. Thus, $u_\infty^+(0,0,1) = u_\infty^-(0,0,-1)$ and this number determines the constant of proportionality with s. □

Finally, the proof of the continuity of $\log h$ follows immediately:

Proof of $\log h \in C(\partial\Omega)$. We note that away from the origin, $\partial\Omega$ is smooth so continuity of the Radon-Nikodym derivative follows from classical potential theory. Furthermore, arguing by symmetry (that is, $-\Omega^+ = \Omega^-$) we have that $\omega^+(B(0,r)) = \omega^-(B(0,r))$ for all $r > 0$. Thus, recalling that $u^\pm$ are the Green's function for $\Omega^\pm$ respectively, we are done if we can show that

$$\lim_{\partial\Omega \ni Q \to 0} \frac{|\nabla u^+|(Q)}{|\nabla u^-|(Q)} = 1.$$

(Recall that where $\partial\Omega$ is smooth, $C^{1,\alpha}$ is sufficient, the Radon-Nikodym derivative is given by the ratio of the derivatives of the Green functions [Kel12]).

Let $Q_i \in \partial\Omega$ with $Q_i \to 0$ and let $|Q_i| = r_i \downarrow 0$. Let $\tilde{u}_i^\pm$ be given by (2.5). Then

$$\frac{\omega^\pm(B_{r_i})}{r_i^2} D\Phi_\theta(r_i x)\nabla \tilde{u}_i^\pm(x) = \nabla u^\pm(\Phi_{-\theta}^{-1}(r_i x)).$$

Let $\tilde{Q}_i = \Phi_\theta(Q_i)/r_i \in \Sigma_s \cap \partial B_1$. We have shown that

$$\frac{|\nabla u^+|(Q_i)}{|\nabla u^-|(Q_i)} = \frac{|D\Phi_\theta(r_i\tilde{Q}_i)\nabla\tilde{u}_i^+(\tilde{Q}_i)|}{|D\Phi_\theta(r_i\tilde{Q}_i)\nabla\tilde{u}_i^-(\tilde{Q}_i)|}.$$

Continuity of $\log h$ follows from Lemma 2.5 (the lemma implies that $\tilde{u}^\pm \to \kappa s$ in $C^{1,\alpha}(\overline{\Omega_s} \cap B_2 \backslash B_{1/2}))$ and the fact that along some subsequence $D\Phi_\theta(r_i x) \to R_{\theta_0}$ for some θ_0 (depending on the subsequence). □

3. The Second Example: Non-Unique Flat Tangents

3.1. **Description and Geometric Properties.** To show non-uniqueness at "flat points" we adapt an example from [Tor94]. We set $\Omega^\pm = \{(x,y,z) \in \mathbb{R}^3 : \pm(z - v(x,y)) > 0\}$, where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined by setting $v(0,0) = 0$,

$$v(x,y) = x \log|\log(r)| \sin(\log|\log(r)|) \quad \text{when } 0 < r = (x^2+y^2)^{1/2} \le 1/100,$$

and smoothly (e.g. $C^{1,\alpha}$) interpolating to $v(x,y) = 1$ when $r \ge 1$.

Lemma 3.1 (see [Tor94, Example 2])**.** *The graph domains $\Omega^\pm$ are chord-arc domains. The interface $\Sigma = \partial\Omega^\pm$ has a continuum of blow-ups at the origin, each of which is a plane $z = mx$ with "slope" $-\infty \le m \le \infty$.*

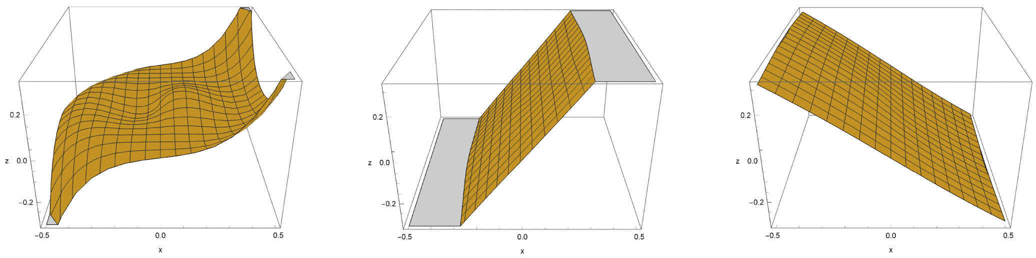

FIGURE 3.1. Blow-ups Σ/r of the interface $\Sigma = \partial\Omega^{\pm}$ of the graph domains. Left: $r = 1$. Center: $r = 10^{-6}$. Right: $r = 10^{-12}$.

Remark 3.2. Moreover, $\Omega^{\pm}$ are vanishing chord-arc domains in the sense of [KT03]. This can be seen as follows. First, every pseudo blow-up (an Attouch-Wets limit Γ of $(\Sigma - Q_i)/r_i$ with $Q_i \to Q$ and $r_i \downarrow 0$) is a plane. Indeed, on the one hand, if $\limsup_{i\to\infty} |Q_i - Q|/r_i = \infty$, then Γ is a plane, because $\Sigma \setminus \{0\}$ is smooth. On the other hand, if $|Q_i|/r_i \leq C$ for all i, then Γ is a translate of a blow-up at Q (see [BL15, Lemma 3.7]), and thus, Γ is a plane by Lemma 3.1. Because every pseudo blow-up is a plane, Σ is locally Reifenberg vanishing. Now, $v \in W^{2,2}(\mathbb{R}^2)$ (see [Tor94]). Hence, by Sobolev embedding, the normal vector of the interface $\hat{n} \in \mathrm{BMO}(\partial\Omega)$ with small BMO norm. Therefore, $\Omega^{\pm}$ are vanishing chord-arc domains; see e.g. [KT97, BEG+22].

3.2. **Potential Theory for the Second Example.** Following the approach of §2.2, we now prove that $\log h \in C(\partial\Omega)$.[2] As before, because $\partial\Omega$ is smooth outside of any neighborhood of the origin, $\log h \in C^{\infty}$ on $\partial\Omega \setminus B_r(0)$ for any $r > 0$. Thus, the key point is to show that $\log h$ is continuous at the origin.

Let $H^{\pm} = \{\pm z > 0\}$ denote the open upper and lower half-spaces. Let $r_i \downarrow 0$ be arbitrary, $K \gg 1$ and write

$$\{z = v(x,y)\} \cap (B_{Kr_i} \backslash B_{r_i/K}) = \Phi^{-1}(\{z = 0\} \cap (B_{Kr_i} \backslash B_{r_i/K})),$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is the homeomorphism given by

$$\Phi(x,y,z) \equiv (x, y, z - v(x,y)). \tag{3.1}$$

Set $\tilde{u}_i^{\pm}(p) = \frac{u^{\pm} \circ \Phi^{-1}(r_i p) r_i}{\omega^{\pm}(B_{r_i}(0))}$, where $u^{\pm}$ are the Green's functions with poles at infinity for $\Omega^{\pm}$, and the $\omega^{\pm}$ are the corresponding harmonic measures. In $H^{\pm} \cap B_K \backslash B_{1/K}$, $\tilde{u}_i^{\pm}$ satisfies

$$-\mathrm{div}(B(r_i p) \nabla \tilde{u}_i^{\pm}(p)) = 0, \qquad B = (\det D\Phi)^{-1}(D\Phi)(D\Phi)^T.$$

Lemma 3.3. *Let $\alpha \in (0,1), K > 1$. The sequence $\tilde{u}_i^{\pm}$ is pre-compact in $C^{1,\alpha}(\overline{H^{\pm}} \cap B_K \backslash B_{1/K})$. Furthermore, there exists a subsequence along which $\tilde{u}_i^{\pm} \to \kappa z_{\pm}$ for some $\kappa > 0$ uniformly on compact subsets of $\mathbb{R}^3$.*

[2]One could prove the weaker result that $\log h \in \mathrm{VMO}(d\omega^{+})$ using Remark 3.2 and standard properties of A_{∞} weights.

Proof. We claim that $\tilde{u}_i^{\pm}$ solves an elliptic PDE with Lipschitz continuous coefficients in $B_K \backslash B_{1/K} \cap H^{\pm}$. Indeed,

(3.2)
$$|B(r_i p) - B(r_i q)| \leq C r_i |p-q| \|DB\|_{L^\infty(B_{Kr_i} \backslash B_{r_i/K})} \overset{\text{[Tor94]}}{\leq} CKr_i \frac{\log|\log(r_i)|}{r_i|\log(r_i)|}|p-q| \leq CK|p-q|,$$

by the fundamental theorem of calculus.

Arguing as in Lemma 2.5 above, $\tilde{u}_i^{\pm}$ are uniformly in $C^{1,\alpha}(\overline{H^+} \cap B_K \backslash B_{1/K})$ for any $\alpha \in (0,1)$ and thus have the desired pre-compactness. Passing to a subsequence and invoking a diagonal argument $\tilde{u}_i^{\pm} \to \tilde{u}_\infty^{\pm}$ uniformly on compacta. Furthermore $\tilde{u}_\infty^{\pm} > 0$ and solves $-\mathrm{div}(B_\infty \nabla \tilde{u}_\infty^{\pm}) = 0$ in $H^{\pm}$ and has $\tilde{u}_\infty^{\pm}(x,y,0) = 0$. We see in (3.2) that B_∞ is constant (as $\log|\log(r_i)|/\log(r_i) \downarrow 0$) and so $-\mathrm{div}(B_\infty \nabla z) = 0$. Again, up to scalar multiplication there is a unique signed solution of $-\mathrm{div}(B_\infty \nabla -) = 0$ in $H^{\pm}$ which vanishes on $\{z = 0\}$ and that has subexponential growth at infinity. Continuing to follow the argument for Lemma 2.5, we conclude that $\tilde{u}_\infty^{\pm} = \kappa_\pm z_\pm$, with $\kappa_+ = \kappa_-$. (Remember that $-\{z > v(x,y)\} = \{z < v(x,y)\}$, because v is odd.) □

Finally, the proof of the continuity of $\log h$ in this context follows exactly as in §2.2 except that we must be more careful estimating $|D\Phi(r_i\tilde{Q}_i)\nabla\tilde{u}^{\pm}(\tilde{Q}_i)|$. (We do not know that $D\Phi(r_i p)$ converges to a rotation as $r_i \downarrow 0$.) However, observe that $\tilde{u}^{\pm} \equiv 0$ on $\{z = 0\}$, so we know that $\nabla\tilde{u}^{\pm}(\tilde{Q}_i)$ is parallel to e_3. Thus, an elementary computation shows that

$$\frac{|D\Phi(r_i\tilde{Q}_i)\nabla\tilde{u}^+(\tilde{Q}_i)|}{|D\Phi(r_i\tilde{Q}_i)\nabla\tilde{u}^-(\tilde{Q}_i)|} = \frac{|\nabla\tilde{u}^+(\tilde{Q}_i)||D\Phi(r_i\tilde{Q}_i)e_3|}{|\nabla\tilde{u}^-(\tilde{Q}_i)||D\Phi(r_i\tilde{Q}_i)e_3|} = \frac{|\nabla\tilde{u}^+(\tilde{Q}_i)|}{|\nabla\tilde{u}^-(\tilde{Q}_i)|}.$$

The quantity on the right hand side converges to 1 by Lemma 3.3. As in §2.2, it follows that $\log h \in C(\partial\Omega)$.

4. Open Questions and Further Directions

We end by presenting some natural open questions. Our first question concerns the size of the set of non-uniqueness:

Question 4.1. Let $\Omega^{\pm} \subset \mathbb{R}^n$ be complementary NTA domains with $\log h \in C(\partial\Omega)$. Is it possible for

$$NU(\Omega) := \{Q \in \partial\Omega : \text{there is no unique (geometric) blow-up at } Q\}$$

to have Hausdorff dimension $n-1$?

We note that a local version of [TT22, Theorem 1.1] implies that the set Γ_1 of flat points in $\partial\Omega$ is uniformly rectifiable. Thus $\omega^{\pm}(NU) = 0 = \mathcal{H}^{n-1}(NU \cap \Gamma_1)$. Further, by the main result of [BET17], $\dim \partial\Omega \backslash \Gamma_1 \leq n-3$. Thus, $\mathcal{H}^{n-1}(NU) = 0$. On the other hand, the example of [AK20] suggests that $\mathcal{H}^{n-2}(NU \cap \Gamma_1) > 0$ may be possible.

The example in §2 (twisted Szulkin) shows that it is possible for all singular points to have non-unique blowups and for the set of singular points with non-unique blowups to have positive $\mathcal{H}^{n-3}$-measure. (When $n \geq 4$, simply take $\Omega^{\pm} \times \mathbb{R}^{n-3}$.) This is sharp by [BET17]. Thus, the natural analogue of Question 4.1 is answered in the affirmative.

Our second question asks what are the possible tangent cones at points of non-unique blow-up:

Question 4.2. Let $C \subset G(n, n-1)$ be a compact, connected subset of the Grassmannian. Does there exist a pair of complementary NTA domains $\Omega^{\pm}$ with $\log h \in C(\partial\Omega)$ and a point $Q \in \partial\Omega$ at which $\mathrm{Tan}(\partial\Omega, Q) = C$?

In §3, we showed that the set $\mathrm{Tan}(\partial\Omega, 0)$ of blow-ups of the interface of the graph domains at the origin consists of all planes $z = mx$ with "slope" $-\infty \leq m \leq +\infty$. For any closed interval $I \subset \mathbb{R}$, it is not hard to adapt the example so that the blowups at the origin are exactly the planes $z = mx$ with $m \in I$. It is known that for any closed set $\Sigma \subset \mathbb{R}^n$ and $Q \in \Sigma$, the set $\mathrm{Tan}(\Sigma, Q)$ of all tangent sets of Σ at Q is closed and connected in the Attouch-Wets topology [BL15]; the statement and proof of this fact was originally motivated by similar statement for tangent measures [Pre87, KPT09].

We may also ask a version of Question 4.2 at points where the blow-ups are homogeneous of higher degree:

Question 4.3. Let $\mathscr{H}_{n,d}$ be the set of degree d homogeneous harmonic polynomials p in $\mathbb{R}^n$ such that $\Omega_p^{\pm} = \{\pm p > 0\}$ are NTA domains. For each $n \geq 3$ and $d \geq 2$ and $C \subset \mathscr{H}_{n,d}$, which is compact and connected, does there exist complementary NTA domains $\Omega^{\pm}$ with $\log h \in C(\partial\Omega)$ and a point $Q \in \partial\Omega$ at which $\mathrm{Tan}(\partial\Omega, Q) = \{\Sigma_p : p \in C\}$?

The condition that $\mathbb{R}^n \setminus \Sigma_p$ is a union of two NTA domains is necessary for Σ_p to arise as a blow-up of the interface of complementary NTA domains. The first step to answering Question 4.3 may be to study the "moduli space" of $\mathscr{H}_{n,d}$ when $d \geq 2$. For example:

Question 4.4. If p and q lie in the same connected component of $\mathscr{H}_{n,d}$, is it true that Σ_q is bi-Lipschitz equivalent to Σ_p?

References

[AK20] Mark Allen and Dennis Kriventsov, *A spiral interface with positive Alt-Caffarelli-Friedman limit at the origin*, Anal. PDE **13** (2020), no. 1, 201–214. MR 4047645

[AKN22] Mark Allen, Dennis Kriventsov, and Robin Neumayer, *Rectifiability and uniqueness of blow-ups for points with positive Alt-Caffarelli-Friedman limit*, preprint, arXiv:2210.03552, 2022.

[AMT20] Jonas Azzam, Mihalis Mourgoglou, and Xavier Tolsa, *A two-phase free boundary problem for harmonic measure and uniform rectifiability*, Trans. Amer. Math. Soc. **373** (2020), no. 6, 4359–4388. MR 4105526

[Bad11] Matthew Badger, *Harmonic polynomials and tangent measures of harmonic measure*, Rev. Mat. Iberoam. **27** (2011), no. 3, 841–870. MR 2895335

[Bad12] Matthew Badger, *Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited*, Math. Z. **270** (2012), no. 1-2, 241–262. MR 2875832

[Bad13] Matthew Badger, *Flat points in zero sets of harmonic polynomials and harmonic measure from two sides*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 111–137. MR 3022709

[BEG$^+$22] S. Bortz, M. Engelstein, M. Goering, T. Toro, and Z. Zhao, *Two phase free boundary problem for Poisson kernels*, Indiana Univ. Math. J. **71** (2022), no. 1, 251–306. MR 4395597

[BET17] Matthew Badger, Max Engelstein, and Tatiana Toro, *Structure of sets which are well approximated by zero sets of harmonic polynomials*, Anal. PDE **10** (2017), no. 6, 1455–1495. MR 3678494

[BET20] Matthew Badger, Max Engelstein, and Tatiana Toro, *Regularity of the singular set in a two-phase problem for harmonic measure with Hölder data*, Rev. Mat. Iberoam. **36** (2020), no. 5, 1375–1408. MR 4161290

[BL15] Matthew Badger and Stephen Lewis, *Local set approximation: Mattila-Vuorinen type sets, Reifenberg type sets, and tangent sets*, Forum Math. Sigma **3** (2015), e24, 63. MR 3482273

[Eng16] Max Engelstein, *A two-phase free boundary problem for harmonic measure*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 4, 859–905. MR 3552015

[GT01] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition. MR 1814364

[Hei06] Juha Heinonen, *What is ... a quasiconformal mapping?*, Notices Amer. Math. Soc. **53** (2006), no. 11, 1334–1335. MR 2268390

[JK82] David S. Jerison and Carlos E. Kenig, *Boundary behavior of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147. MR 676988

[Kel12] O. D. Kellogg, *Harmonic functions and Green's integral*, Trans. Amer. Math. Soc. **13** (1912), no. 1, 109–132. MR 1500909

[KPT09] C. Kenig, D. Preiss, and T. Toro, *Boundary structure and size in terms of interior and exterior harmonic measures in higher dimensions*, J. Amer. Math. Soc. **22** (2009), no. 3, 771–796. MR 2505300

[KT97] Carlos E. Kenig and Tatiana Toro, *Harmonic measure on locally flat domains*, Duke Math. J. **87** (1997), no. 3, 509–551. MR 1446617

[KT99] Carlos E. Kenig and Tatiana Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454. MR 1726699

[KT03] Carlos E. Kenig and Tatiana Toro, *Poisson kernel characterization of Reifenberg flat chord arc domains*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 3, 323–401. MR 1977823

[KT06] Carlos Kenig and Tatiana Toro, *Free boundary regularity below the continuous threshold: 2-phase problems*, J. Reine Angew. Math. **596** (2006), 1–44. MR 2254803

[Mat95] Pertti Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995, Fractals and rectifiability. MR 1333890

[Pre87] David Preiss, *Geometry of measures in* $\mathbf{R}^n$*: distribution, rectifiability, and densities*, Ann. of Math. (2) **125** (1987), no. 3, 537–643. MR 890162

[PT20] Martí Prats and Xavier Tolsa, *The two-phase problem for harmonic measure in VMO*, Calc. Var. Partial Differential Equations **59** (2020), no. 3, Paper No. 102, 58. MR 4102355

[Szu79] Andrzej Szulkin, *An example concerning the topological character of the zero-set of a harmonic function*, Math. Scand. **43** (1978/79), no. 1, 60–62. MR 523825

[Tor94] Tatiana Toro, *Surfaces with generalized second fundamental form in* L^2 *are Lipschitz manifolds*, J. Differential Geom. **39** (1994), no. 1, 65–101. MR 1258915

[TT22] Xavier Tolsa and Tatiana Toro, *The two-phase problem for harmonic measure in VMO and the chord-arc condition*, preprint, arXiv:2209.14346, 2022.

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