Harmonic Polynomials and Free Boundary Regularity for Harmonic Measure from Two Sides

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Abstract

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We use tools from geometric measure theory to catalog fine behavior of harmonic measure on a class of two-sided domains $\Omega \subset \mathbb{R}^n$ in n-dimensional Euclidean space, with $n \geq 3$. Assume the interior $\Omega^+ = \Omega$ and exterior $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ of Ω are NTA domains, equipped with harmonic measures ω^+ and ω^-, respectively. We prove that if ω^+ and ω^- are mutually absolutely continuous and the logarithm of their Radon-Nikodym derivative $d\omega^-/d\omega^+$ has vanishing mean oscillation, then the boundary $\partial\Omega$ can be written as a finite disjoint union of sets Γ_k $(1 \leq k \leq d)$ with the following properties. For each $Q \in \Gamma_k$, every blow-up of $\partial\Omega$ centered at Q is the zero set of a homogeneous harmonic polynomial of degree k which separates space into two connected components; the set Γ_1 of "flat points" is relatively open and locally Reifenberg flat with vanishing constant; and the set $\Gamma_2 \cup \cdots \cup \Gamma_d$ of "singularities" has harmonic measure zero.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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I first learned about harmonic measure from Don Marshall during a topics course in Autumn 2007. It has captivated me ever since and I am grateful for his introduction.

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Let me add that my parents have always supported me—through trials and triumphs. To my Mom, thank you for being an unfailing advocate and for instilling my faith. And to my Dad, thank you for leading me to pursue work I enjoy.

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DEDICATION

To Christine Betlem

and George Reuter …

Teachers who built up my

mathematical foundations

Chapter 1

INTRODUCTION

...the Dirichlet problem has recently been through a period of remarkable development. But it is characteristic of scientific progress that each advance raises new questions.

— O. D. Kellogg, 1926.

1.1 Harmonic Measure

The search for harmonic functions in a bounded domain with prescribed boundary values leads to a distinguished family of probability measures on the boundary of the domain. These measures—called harmonic measures—exist on any boundary, fractal or smooth, and encode regularity of the boundary on nice domains. We recall their construction via Perron's solution of the Dirichlet problem.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Recall a real-valued function $u : \Omega \to \mathbb{R}$ is *harmonic* in Ω if $u \in C^2(\Omega)$ and u solves *Laplace's equation*,

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0. \tag{1.1}$$

The classical *Dirichlet problem* is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

for every $f \in C(\partial\Omega)$. However, there exist domains (see Example 1.1) for which (1.2) does not admit a solution for every continuous boundary data f. To overcome this difficulty, Perron [33] suggested that instead of requiring $u = f$ on $\partial\Omega$, one should look for harmonic functions on Ω which best approximate f on $\partial\Omega$ from above and below. Given a real-valued function $f : \partial\Omega \to \mathbb{R}$, define the *upper class* U_f of f by

$$U_f = \Big\{ u : \Omega \to \mathbb{R} \cup \{+\infty\} \mid u \equiv +\infty \text{ on } \Omega \text{ or}, \quad \Delta u \leq 0 \text{ and } \liminf_{X \to Q} u(X) \geq f(Q) \text{ for all } Q \in \partial\Omega \Big\}, \tag{1.3}$$

and define the *lower class* L_f of f by

$$L_f = \Big\{ u : \Omega \to \mathbb{R} \cup \{-\infty\} \mid u \equiv -\infty \text{ on } \Omega \text{ or}, \quad \Delta u \geq 0 \text{ and } \limsup_{X \to Q} u(X) \leq f(Q) \text{ for all } Q \in \partial\Omega \Big\}. \tag{1.4}$$

We say f is *resolutive* if $Hf(X) = \inf\{u(X) : u \in U_f\} = \sup\{u(X) : u \in L_f\}$ for all $X \in \Omega$. When it exists, the function Hf is harmonic and is called the *Perron solution* of the Dirichlet problem with data f. If u is a classical solution of the Dirichlet problem (1.2) with data f, then u coincides with the Perron solution Hf.

Example 1.1. Let $\Omega = \{X \in \mathbb{R}^2 : 0 < |X| < 1\}$ denote the punctured unit disk. Consider the function $f : \partial\Omega \to \mathbb{R}$ defined by

$$f(X) = \begin{cases} 0, & \text{if } |X| = 1 \\ 1 & \text{if } X = (0,0) \end{cases} \tag{1.5}$$

Observe the function f is continuous on $\partial\Omega$, but the Dirichlet problem on Ω does not admit a classical solution with boundary data f. To see why, suppose there exists $u \in C^2(\Omega) \cap C(\overline{\Omega})$ which solves (1.2). Applying the maximum principle (for a course on basic properties of harmonic functions such as the maximum principle and the mean value property, see [2]) to the subharmonic function $u_\varepsilon(X) = u(X) + \varepsilon \log |X|$ yields $u_\varepsilon(X) \leq 0$ for all $X \in \Omega$, or equivalently, $u(X) \leq -\varepsilon \log |X|$ for all $X \in \Omega$. Letting $\varepsilon \to 0$ forces $u(X) \leq 0$ for all $X \in \Omega$. Since $u \in C(\overline{\Omega})$, we find $u(0,0) \leq 0$, as well. This contradicts the assumption $u = f$ on $\partial\Omega$. Nevertheless the function f is resolutive. Because the upper class U_f contains $-\varepsilon \log |X|$ for all $\varepsilon > 0$ and the lower class L_f contains the zero function, the Perron solution of the Dirichlet problem on Ω with boundary data f exists and is $Hf \equiv 0$.

Returning to the general setup, Wiener [39] showed that on a bounded domain every continuous function $f \in C(\partial\Omega)$ is resolutive. Moreover, by the maximum principle for harmonic functions, the Perron solution of the Dirichlet problem satisfies

$$H(c_1 f + c_2 g) = c_1 Hf + c_2 Hg \quad \text{for all } f, g \in C(\partial\Omega) \text{ and } c_1, c_2 \in \mathbb{R} \tag{1.6}$$

and

$$c_1 \leq f \leq c_2 \Longrightarrow c_1 \leq Hf \leq c_2 \quad \text{for all } f \in C(\partial\Omega) \text{ and } c_1, c_2 \in \mathbb{R}. \tag{1.7}$$

Thus, for each $X \in \Omega$, the rule $f \mapsto Hf(X)$ describes a positive linear functional on $C(\partial\Omega)$ with operator norm 1. By the Riesz representation theorem, for each $X \in \Omega$, there exists a unique Borel regular probability measure ω_Ω^X supported on $\partial\Omega$ such that

$$Hf(X) = \int_{\partial\Omega} f d\omega_\Omega^X \quad \text{for all } f \in C(\partial\Omega). \tag{1.8}$$

Following Nevanlinna [32], we call ω_Ω^X the *harmonic measure* of Ω with *pole* at $X \in \Omega$.

Example 1.2. Let $\mathbb{B}^n$ denote the open unit ball in $\mathbb{R}^n$ ($n \geq 2$) with volume ω_n. If $X \in \mathbb{B}^n$, then the harmonic measure of $\mathbb{B}^n$ with pole at X is given by

$$\omega_{\mathbb{B}^n}^X(E) = \int_E \frac{1 - |X|^2}{|X - Q|^n} \frac{d\mathcal{H}^{n-1}(Q)}{n\omega_n} \quad \text{for every Borel set } E \subset S^{n-1} \tag{1.9}$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure (Definition 2.1), normalized to agree with surface measure on $S^{n-1} = \partial\mathbb{B}^n$. In particular, at the origin,

$$\omega_{\mathbb{B}^n}^0(E) = \frac{\mathcal{H}^{n-1}(E)}{n\omega_n}. \tag{1.10}$$

Example 1.3. Let $\Omega \subset \mathbb{R}^2$ be a simply connected planar domain, bounded by a Jordan curve. Given $X \in \Omega$, let $f : \mathbb{B}^2 \to \Omega$ be a Riemann conformal map from the open unit disk $\mathbb{B}^2$ onto Ω such that $f(0) = X$. Then the harmonic measure of Ω with pole at X is given by

$$\omega_\Omega^X(E) = \omega_{\mathbb{B}^2}^0(f^{-1}(E)) = \frac{\mathcal{H}^1(f^{-1}(E))}{2\pi} \tag{1.11}$$

for every Borel set $E \subset \partial\Omega$ where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, normalized to agree with the length of a set.

Example 1.4. Assume that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain of class C^2. Let $X \in \Omega$ and let $G^X = G(\cdot, X)$ be the Green function of Ω with pole at X (for example, see [19]). Using Green's identities, one can show that

$$\omega_\Omega^X(E) = -c_n \int_E \frac{dG^X}{d\vec{N}_Q}(Q)\, d\sigma(Q) \tag{1.12}$$

for every Borel set $E \subset \partial\Omega$, where $\vec{N}_Q$ denotes the outer unit normal vector at $Q \in \partial\Omega$, $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ denotes surface measure on $\partial\Omega$, and the constant $c_n > 0$ depends only on n. The Radon-Nikodym derivative of harmonic measure ω_Ω^X with respect to surface measure

$$k_\Omega^X(Q) := \frac{d\omega_\Omega^X}{d\sigma}(Q) = -c_n \frac{dG^X}{d\vec{N}_Q}(Q) \tag{1.13}$$

is called the *Poisson kernel* of Ω with pole at X.

In [6], Dahlberg showed that (1.12) and (1.13) persist on Lipschitz domains, provided that one interprets $dG^X/d\vec{N}_Q$ as a non-tangential limit defined only at $\mathcal{H}^{n-1}$-a.e. $Q \in \partial\Omega$.

Remarks.

1.5 If $X_1, X_2 \in \Omega$, then Harnack's inequality for positive harmonic functions implies that the harmonic measures $\omega_\Omega^{X_1}$ and $\omega_\Omega^{X_2}$ are *mutually absolutely continuous*, i.e.

$$\omega_\Omega^{X_1}(E) = 0 \iff \omega_\Omega^{X_2}(E) = 0 \quad \text{for every Borel set } E \subset \partial\Omega. \tag{1.14}$$

Thus, by a small abuse of notation, one may consider "the harmonic measure" ω_Ω ($= \omega_\Omega^{X_0}$) of a domain Ω (with respect to some fixed, but unspecified $X_0 \in \Omega$).

1.6 If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then $\omega_\Omega \ll \mathcal{H}^{n-2}$. That is, if $E \subset \partial\Omega$ is a Borel set and $\mathcal{H}^{n-2}(E) = 0$, then $\omega_\Omega(E) = 0$. See Appendix A.

1.7 If $\Omega \subset \mathbb{R}^n$ is an unbounded domain such that $\partial\Omega$ has positive logarithmic capacity (when $n = 2$) or positive Newtonian capacity (when $n \geq 3$), then harmonic measure of Ω with pole at $X \in \Omega$ can be similarly defined provided one replaces the condition $f \in C(\partial\Omega)$ in (1.8) with $f \in C_b(\partial\Omega)$, the space of bounded continuous functions. See Helms [13] for details.

1.8 In [16], Kakutani gave a purely probabilistic interpretation of harmonic measure. On a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the harmonic measure $\omega_\Omega^X(E)$ describes the probability that Brownian motion started at $X \in \Omega$ first exits the domain through the set $E \subset \partial\Omega$. However, this approach to harmonic measure plays no role in the sequel.

As Example 1.3 illustrates, the study of harmonic measure on planar domains $(n = 2)$ is tied up with the theory of conformal maps. For further information about harmonic measure in two dimensions, we refer the reader to the monograph by Garnett and Marshall [11]. Because higher dimensions $(n \geq 3)$ lack non-trivial conformal maps, one needs to find different tools to study harmonic measure in space. Below we use elements of geometric measure theory to study a free boundary regularity problem for harmonic measure.

1.2 Free Boundary Problems for Harmonic Measure

There is a relationship between regularity of harmonic measure and regularity of the domain. A rough heuristic valid on sufficiently smooth domains is that the boundary of a domain should have one derivative higher regularity than the Poisson kernel. For example, in one direction Kellogg [19] proved that if $\partial\Omega$ can be represented locally as the graph of a $C^{1,\alpha}$ function for some $\alpha \in (0,1)$, then the Radon-Nikodym derivative $d\omega_\Omega/d\sigma$ of harmonic measure with respect to surface measure is $C^{0,\alpha}$ as a function of $\partial\Omega$. One may ask if the converse also holds. If we know that the harmonic measure (supported on the boundary) has good properties, but not *a priori* what the boundary looks like, then we can ask what properties must the boundary enjoy. This is a free boundary problem for harmonic measure:

Free Boundary Problem 1. Let $\Omega \subset \mathbb{R}^n$ be a domain of *locally finite perimeter* equipped with harmonic measure ω_Ω and surface measure $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$. If the *Poisson kernel* $k_\Omega = d\omega_\Omega/d\sigma$ is sufficiently regular, how regular is the boundary $\partial\Omega$ of the domain Ω?

Let us recall a few results about Free Boundary Problem 1, in the historical order that they appeared. The first result concerns higher regularity of the domain.

Theorem 1.9 (Kinderlehrer and Nirenberg [25]). *Assume that $\Omega \subset \mathbb{R}^n$ is a C^1 domain.*

1. *If $\log k_\Omega \in C^{1+m,\alpha}$ for some $m \geq 0$ and $\alpha \in (0,1)$, then $\partial\Omega$ is $C^{2+m,\alpha}$.*

2. *If $\log k_\Omega \in C^\infty$, then $\partial\Omega$ is C^∞.*

3. *If $\log k_\Omega$ is real analytic, then $\partial\Omega$ is real analytic.*

Remark 1.10. In Theorem 1.9 and in further results below, one assumes some condition on the logarithm of the Poisson kernel rather than just the Poisson kernel. This is shorthand, which just says that the Poisson kernel satisfies a certain condition *and* the Poisson kernel is strictly positive and finite.

The next result answers Free Boundary Problem 1 at the scale of Hölder continuity. Theorem 1.12 also weakens the assumption from Theorem 1.9 that $\Omega \subset \mathbb{R}^n$ is at least C^1; instead it requires that (i) Ω has surface measure with uniform positive and finite density and (ii) $\partial\Omega$ is locally sufficiently "flat".

Definition 1.11. A domain $\Omega \subset \mathbb{R}^n$ of locally finite perimeter is *Ahlfors regular* if there exist constants $C_1, C_2 > 0$ and $r_0 > 0$ such that $C_1 r^{n-1} \leq \mathcal{H}^{n-1}(\partial\Omega \cap B(Q,r)) \leq C_2 r^{n-1}$ for all $Q \in \partial\Omega$ and for all $r \in (0, r_0)$.

Theorem 1.12 (Alt and Caffarelli [1]). *Assume that (i) $\Omega \subset \mathbb{R}^n$ is Ahlfors regular and (ii) for every $Q \in \partial\Omega$ there exists $r_Q > 0$ such that $\partial\Omega \cap B(Q, r_Q)$ is flat, in the sense that $\theta_{\partial\Omega}(Q, r_Q) < \delta$ for some $\delta < \delta_n$ (see Definition 2.8). If $\log k_\Omega \in C^{0,\beta}$ for some $\beta \in (0,1)$, then Ω is of class $C^{1,\alpha}$ for some $\alpha = \alpha(\beta) > 0$.*

Two improvements of Theorem 1.12 subsequently appeared.

Theorem 1.13 (Caffarelli [5]). *In Theorem 1.12, one can replace the hypotheses (i)–(ii) with the assumption that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain.*

Theorem 1.14 (Jerison [14]). *In Theorem 1.12, $\alpha = \beta$.*

One may also ask about the limiting behavior of Theorem 1.12 when $\alpha \to 0$. In [14] Jerison also proves that if $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and $\log k_\Omega \in C^0$ is continuous, then $\partial\Omega$ can be locally represented as the graph of a function u whose distributional gradient $\nabla u \in \mathrm{VMO}$ has *vanishing mean oscillation* (see [37]). It is possible to obtain still further results under weaker conditions on the domain and its Poisson kernel. For example, in [23] Kenig and Toro study Free Boundary Problem 1 on the class of Reifenberg flat chord arc domains (which are not necessarily locally representable as the graph of a function) whose Poisson kernel satisfy the condition $\log k_\Omega \in \mathrm{VMO}$.

A possible defect in the statement of Free Boundary Problem 1 is that a typical domain on which harmonic measure is defined (i.e. any bounded domain) does not possess a well-defined notion surface measure. Thus one would like to find a suitable replacement for the Poisson kernel in order to gauge "regularity of harmonic measure" on more general domains. One idea is to compare the harmonic measure ω_Ω on Ω with the harmonic measure $\omega_{\mathbb{R}^n\setminus\overline{\Omega}}$ on the *exterior* of Ω, which makes sense as long as $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and $\partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial\Omega$.

Definition 1.15. We say that $\Omega \subset \mathbb{R}^n$ is a *two-sided domain* if (i) $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ are connected open sets and (ii) $\partial\Omega^+ = \partial\Omega^-$.

Free Boundary Problem 2. Let $\Omega \subset \mathbb{R}^n$ be a two-sided domain with harmonic measure ω^+ on Ω^+ and harmonic measure ω^- on Ω^-. If the *two-sided kernel* $f_\Omega = d\omega^-/d\omega^+$ is sufficiently regular, how regular is the boundary $\partial\Omega$ of the domain Ω?

Unfortunately simple examples (for instance, see Example 1.21 below) show that $\partial\Omega$ may not be smooth even if $f_\Omega \equiv 1$. Nevertheless the following positive result is known. Under certain mild conditions on the domain (see §5.1 for the definition of NTA domain) and weak regularity of $\log f_\Omega$ below the continuous threshold, Kenig and Toro [24] are able to identify blow-ups of $\partial\Omega$ in the Hausdorff distance (see Definition 2.6). In other words, they identify the shape of the boundary that one sees when zooming in on a point of the boundary: every blow-up of the boundary is the *zero set* of a *harmonic polynomial*.

Theorem 1.16 (Kenig and Toro [24]). *Suppose that* $\Omega \subset \mathbb{R}^n$ *is a 2-sided NTA domain. If* $\omega^+ \ll \omega^- \ll \omega^+$ *and* $\log f_\Omega \in \mathrm{VMO}(d\omega^+)$*, then for every* $Q \in \partial\Omega$ *and sequence* $r_i \downarrow 0$*, there exists a subsequence of* r_i *(which we relabel) and a harmonic polynomial* $h : \mathbb{R}^n \to \mathbb{R}$ *with* $h(0) = 0$ *such that*

$$\begin{aligned} \frac{\Omega^\pm - Q}{r_i} &\to \Omega_h^\pm \quad \text{in the Hausdorff distance uniformly on compact sets}, \\ \frac{\partial\Omega - Q}{r_i} &\to \partial\Omega_h^\pm \quad \text{in the Hausdorff distance uniformly on compact sets}, \end{aligned} \tag{1.15}$$

where $\Omega_h^\pm = \{x \in \mathbb{R}^n : h^\pm(x) > 0\}$ *is a 2-sided NTA domain whose boundary* $\partial\Omega_h^+ = \partial\Omega_h^-$ *is the zero set of* h*. Moreover, if* $\partial\Omega$ *is* δ*-Reifenberg flat (Definition 2.11) for some* $\delta < \delta_n$*, then* $\partial\Omega$ *is Reifenberg flat with vanishing constant (Definition 2.13).*

1.3 Results and Organization

In this dissertation, our goal is to provide a complete description of the free boundary $\partial\Omega$ under the assumptions on $\Omega^\pm$ and $\omega^\pm$ in Theorem 1.16. We show that blow-ups of the boundary are homogeneous, i.e. only the zero sets of *homogeneous* harmonic polynomials appear in (1.15). Moreover, we show that the degree of the polynomial h appearing in (1.15) is uniquely determined at each $Q \in \partial\Omega$.

Main Theorem (Structure Theorem for Free Boundary Problem 2). *Let* $\Omega^+ = \Omega \subset \mathbb{R}^n$ *and* $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ *be NTA domains, equipped with harmonic measures* ω^+ *and* ω^-*, respectively. If* $\omega^+ \ll \omega^- \ll \omega^+$ *and* $\log f_\Omega \in \mathrm{VMO}(d\omega^+)$*, then we can write the boundary* $\partial\Omega$ *of* $\Omega^\pm$ *as a finite disjoint union,*

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d \tag{1.16}$$

where $d \geq 1$ *only depends on the dimension* n *and on the NTA constants of* $\Omega^\pm$ *and the sets* Γ_k *(*$1 \leq k \leq d$*) have the following properties:*

1. *Every blow-up of* $\partial\Omega$ *centered at* $Q \in \Gamma_k$ *(see Definition 2.17) is the zero set of a homogeneous harmonic polynomial of degree* k *separating* $\mathbb{R}^n$ *into two components.*

2. *The set of "flat points" Γ_1 is relatively open in $\partial\Omega$ and Γ_1 is locally Reifenberg flat with vanishing constant (see Definition 2.14).*

3. *The set of "singularities" $\Gamma_2 \cup \cdots \cup \Gamma_d$ is closed and has harmonic measure zero.*

Remark 1.17. In the Main Theorem "separating $\mathbb{R}^n$ into two components" means that if the zero set $\Sigma_h = \{x \in \mathbb{R}^n : h(x) = 0\}$ of a homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ is a blow-up of $\partial\Omega$ then the open set $\mathbb{R}^n \setminus \Sigma_h = \{x \in \mathbb{R}^n : h(x) \neq 0\}$ has exactly two connected components. This condition on Σ_h is an immediate consequence of the assertion in Theorem 1.16 that $\Omega_h^\pm = \{h^\pm > 0\}$ is a 2-sided NTA domain. Alternatively, see the proof of Theorem 5.19.

The existence of polynomials with this separation property depends on the dimension. When $n = 2$, the zero set Σ_h separates $\mathbb{R}^2$ into two components if and only if $\deg h = 1$. When $n = 3$, the zero set Σ_h can separate $\mathbb{R}^3$ into two components only if $\deg h$ is odd. Thus the separation condition on Σ_h restricts the existence of the sets Γ_k in low dimensions. For further discussion on the number of components of $\mathbb{R}^n \setminus \Sigma_h$, see Chapter 3.

Corollary 1.18. *If $\Omega \subset \mathbb{R}^2$ satisfies the hypotheses of the Main Theorem, then $\partial\Omega = \Gamma_1$.*

Corollary 1.19. *If $\Omega \subset \mathbb{R}^3$ satisfies the hypotheses of the Main Theorem, then $d \geq 1$ is odd and $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \cdots \cup \Gamma_d$.*

Remark 1.20. The Main Theorem asserts that the set Γ_1 of flat points in the free boundary is actually locally Reifenberg flat with vanishing constant. As an immediate consequence, we know that Γ_1 has Hausdorff dimension $n - 1$ (see Corollary 2.16).

Example 1.21. Consider the homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ ($n \geq 3$) given by

$$h(X) = X_1^2(X_2 - X_3) + X_2^2(X_3 - X_1) + X_3^2(X_1 - X_2) - X_1X_2X_3. \tag{1.17}$$

Then the domain $\Omega = \{X \in \mathbb{R}^n : h(X) > 0\}$ is a 2-sided NTA domain; in particular, $\partial\Omega = \{X \in \mathbb{R}^n : h(X) = 0\} = \Sigma_h$ separates $\mathbb{R}^n$ into two components (see Figure 3.3).

If ω^+ and ω^- denote the interior and exterior harmonic measure on Ω with pole at infinity, then $\omega^+ = \omega^-$, $\log f_\Omega \equiv 0$ and

$$\frac{\partial\Omega - X}{r} = \Sigma_h \quad \text{for all } X = (0,0,0,X_4,\dots,X_n) \text{ and } r > 0. \tag{1.18}$$

In particular, Σ_h is a blow-up of $\partial\Omega$ at the origin. Thus $0 \in \Gamma_3$ and non-planar blow-ups of the boundary can appear even when $\log f_\Omega$ is analytic. Moreover, for all $n \geq 3$ it is possible for the set "singularities" $\partial\Omega \setminus \Gamma_1 = \Gamma_2 \cup \dots \cup \Gamma_d$ to have Hausdorff dimension $\geq n-3$.

An obvious modification of the domain Ω in Example 1.21 shows that:

Proposition 1.22. *The zero set Σ_h of a harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ appears in (1.15) as a blow-up of $\partial\Omega$ for some 2-sided NTA domain $\Omega \subset \mathbb{R}^n$ if and only if h is homogeneous and Σ_h separates $\mathbb{R}^n$ into two components.*

Proof. Necessity was established by the Main Theorem, so it remains to check sufficiency. Let $h : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous harmonic polynomial such that Σ_h separates $\mathbb{R}^n$ into two components. Then $\Omega = \{X \in \mathbb{R}^n : h(X) > 0\}$ is a 2-sided NTA domain which satisfies the hypotheses of the Main Theorem. Moreover, since

$$\frac{\partial\Omega}{r} = \Sigma_h \quad \text{for all } r > 0, \tag{1.19}$$

Σ_h is the unique blow-up of $\partial\Omega = \Sigma_h$ at the origin. □

1.3.1 Proof Strategy

The method in [24] relates the geometric blow-ups of the boundary to *tangent measures* (Definition 2.26) of the harmonic measure. Thus information about the free boundary may be obtained by studying tangent measures of harmonic measure. This is the strategy that we employ below: to identify the polynomials appearing in (1.15), we study properties of certain "polynomial harmonic measures" (see Definition 4.1) in the topology of weak convergence of Radon measures of $\mathbb{R}^n$. In Theorem 5.15, we establish a "self-improving" property of the tangent measures of harmonic measure. We prove that on an NTA domain $\Omega \subset \mathbb{R}^n$ if every tangent measure $\nu \in \mathrm{Tan}(\omega_\Omega, Q)$ to the harmonic measure ω_Ω at a point $Q \in \partial\Omega$ is a polynomial harmonic measure associated to a polynomial of degree at most d, then we automatically know every tangent measure is associated to a *homogeneous* harmonic polynomial of degree k, for some $1 \leq k \leq d$. We then invoke the correspondence between tangent measures of harmonic measure and blow-ups of the boundary on an NTA domain to establish Part 1 of the Main Theorem (see Theorem 5.19).

The proof of the Theorem 5.15 illustrates the versatility of a powerful technique from geometric measure theory. Tangent measures are a tool that encode information about the support of a measure, similar to how derivatives describe the local behavior of functions. A remarkable feature is that under general conditions (Theorem 2.35) the cone of tangent measures at a point is connected. This fact lies at the core of Preiss' celebrated paper on rectifiability [34] and recently enabled Kenig, Preiss and Toro [20] to compute the upper Hausdorff dimension of harmonic measure on 2-sided NTA domains whose interior and exterior harmonic measures are mutually absolutely continuous. To appreciate the second result the reader should compare Theorem 1.23 with the dimension of harmonic measure on Wolff snowflakes studied in Wolff [40] and Lewis, Verchota and Vogel [26].

Theorem 1.23 ([20] Theorem 4.3)**.** *Assume $\Omega \subset \mathbb{R}^n$ is 2-sided NTA. If $\omega^+ \ll \omega^- \ll \omega^+$, then $\overline{\dim}_H\, \omega^+ = \overline{\dim}_H\, \omega^- = n-1$ (see Definition 2.4).*

In previous instances, connectedness of the cone of tangent measures at a point was used to conclude that the tangent measures, of a certain measure μ, at *almost every* point x in the support of μ, belong to the cone $\mathcal{F}_1$ of flat measures, i.e. the collection of Hausdorff measures restricted to hyperplanes through the origin. The authors in [20] express an opinion that the connectedness of tangent measures "should be useful in other situations where questions of size and structure of the support of a measure arise." Below, in the proof of Theorem 5.15, we use the connectedness of the cone of tangent measures to show that tangent measures of harmonic measure ω_Ω, at *every* $Q \in \partial\Omega$, live in a cone of measures $\mathcal{F}_k$ associated to homogeneous harmonic polynomials of degree k. This is the first use of this technique to catalog “irregular” tangent structures. We expect further applications of the connectedness of the cone of tangent measures to exist in other settings.

The proof of Part 2 of the Main Theorem is based on a special property of the zero sets of harmonic polynomials, established in Chapter 3. We demonstrate (see Theorem 3.2) that at every point of the zero set Σ_h of a nonconstant harmonic polynomial h, either the zero set becomes arbitrarily flat on small scales or the zero set stays far away from a plane at every scale. The first alternative corresponds exactly with regular points of the zero set; the second alternative corresponds exactly with singularities of the polynomial. Moreover, these statements are quantitative and uniform across all harmonic polynomials of degree at most $d \geq 1$. Applying Theorem 3.2 to the situation in the Main Theorem, we are able to show that if $\partial\Omega \cap B(Q,r)$ is sufficiently flat at some initial scale r, then $\partial\Omega \cap B(Q,s)$ remains flat on a smaller scale $s < r$. Proceeding by induction, we then show that every blow-up of $\partial\Omega$ at Q is flat provided that $\partial\Omega \cap B(Q,r)$ is sufficiently flat at a single scale. This lets us conclude that membership in Γ_1 is an open condition (see Theorem 5.23) and Γ_1 is locally Reifenberg flat (see Corollary 5.24).

The proof of Part 3 of the Main Theorem is based on a property of tangent measures: for any Radon measure μ the cone of tangent measures $\mathrm{Tan}(\mu, x)$ of μ at $x \in \operatorname{spt}\mu$ is *translation invariant* for μ-a.e. x (see Theorem 2.29 and Definition 2.31). We can prove (see Proposition 5.26) that the cone $\mathrm{Tan}(\omega^{\pm}, Q)$ of tangent measures of harmonic measure $\omega^{\pm}$ at $Q \in \partial\Omega$ fails to be translation invariant at every $Q \in \Gamma_2 \cup \cdots \cup \Gamma_d$. Thus we conclude the set of singularities $\Gamma_2 \cup \cdots \cup \Gamma_d$ has harmonic measure zero.

1.3.2 Organization

There are four additional chapters and one appendix. In *Chapter 2*, we review elements of geometric measure theory used extensively throughout the sequel. In *Chapter 3*, we study zero sets of harmonic polynomials. In particular, we establish a dichotomy between the “local flatness” of the zero set of a harmonic polynomial at regular points of the polynomial (where Σ_h is arbitrarily flat) and at singularities of the polynomial (where Σ_h is far from flat). In *Chapter 4*, we study properties of polynomial harmonic measures in the topology of weak convergence of Radon measures. And, in *Chapter 5*, we combine results from Chapters 2–4 with the previous work of Kenig and Toro [24] to prove the Main Theorem. The *Appendix* is a self-contained proof a lower bound for the dimension of harmonic measure on any domain.

Chapter 2

ESSENTIAL GMT

In this chapter we gather together ingredients from Geometric Measure Theory (GMT), collecting the definitions, notations and tools used throughout the sequel. Absent further comment, all sets and measures live in n-dimensional Euclidean space $\mathbb{R}^n$.

2.1 Dimension of Sets and Measures

Let $B(x, r)$ denote the closed ball with center $x \in \mathbb{R}^n$ and radius $r > 0$. We abbreviate $B_r = B(0, r)$ for all $r > 0$. Note that $\partial B_1 = S^{n-1}$, the unit sphere in $\mathbb{R}^n$. The unit ball B_1 has volume ω_n; the unit sphere S^{n-1} has surface measure $\sigma_{n-1} = n\omega_n$.

Definition 2.1. For any $s \geq 0$, the *s-dimensional Hausdorff measure* $\mathcal{H}^s$ is the Borel regular outer measure on $\mathbb{R}^n$ defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \inf \left\{ c_s \sum_{i=1}^{\infty} \operatorname{diam}(E_i)^s : E \subset \bigcup_{i=1}^{\infty} E_i \text{ and } \operatorname{diam} E_i \leq \delta \right\} \tag{2.1}$$

for all $E \subset \mathbb{R}^n$, where

$$c_s = \frac{\pi^{s/2}}{2^s \Gamma(s/2 + 1)}. \tag{2.2}$$

The constant c_s is chosen so that n-dimensional Hausdorff measure $\mathcal{H}^n$ agrees with the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$ [10]. Notice that the sets E_i covering E in (2.1) are arbitrary. Instead if we require the sets E_i covering E to be balls, then we obtain a different measure that is comparable to Hausdorff measure.

Definition 2.2. The *s-dimensional spherical measure* $\mathcal{S}^s$ is the Borel regular outer measure on $\mathbb{R}^n$ defined by

$$\mathcal{S}^s(E) = \lim_{\delta \to 0} \inf \left\{ c_s \sum_{i=1}^{\infty} (2r_i)^s : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \text{ and } 2r_i \leq \delta \right\} \tag{2.3}$$

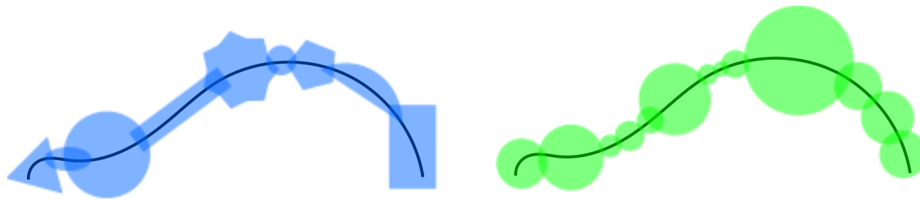

Figure 2.1: Hausdorff Measure versus Spherical Measure

The Hausdorff and spherical measures are related to each other by the inequalities

$$\mathcal{H}^s(E) \leq \mathcal{S}^s(E) \leq 2^s \mathcal{H}^s(E) \quad \text{for all } E \subset \mathbb{R}^n. \tag{2.4}$$

In particular, a set $E \subset \mathbb{R}^n$ is a $\mathcal{H}^s$-null set if and only if E is a $\mathcal{S}^s$-null set. The proof of (2.4) is a straight-forward exercise and is valid in any metric space. With additional work, one can improve the upper bound for subsets of Euclidean space.

Lemma 2.3 ([10] Corollary 2.10.42)**.** $\mathcal{S}^s(E) \leq \left(\frac{2n}{n+1}\right)^{s/2} \mathcal{H}^s(E)$ *for all* $E \subset \mathbb{R}^n$.

Computing the exact value of the Hausdorff measure of a set is impossible in practice. Instead one would like to decide whether the s-dimensional Hausdorff measure of a set is zero, positive and finite, or infinite. If $E \subset \mathbb{R}^n$ and $\mathcal{H}^s(E) < \infty$ for some $s \geq 0$ then $\mathcal{H}^t(E) = 0$ for all $t > s$. Equivalently, if $\mathcal{H}^t(E) > 0$ for some $t > 0$, then $\mathcal{H}^s(E) = \infty$ for all $0 \leq s < t$. Together these facts imply that for any set $E \subset \mathbb{R}^n$ there is at most one $s \geq 0$ such that $0 < \mathcal{H}^s(E) < \infty$. This justifies the following definition of the dimension of a set.

Definition 2.4. The *Hausdorff dimension* $\dim_H E$ of a set $E \subset \mathbb{R}^n$ is the unique number $d \in [0, n]$ such that $\mathcal{H}^s(E) = \infty$ for all $s < d$ and $\mathcal{H}^t(E) = 0$ for all $t > d$.

Measures have two candidates for Hausdorff dimension. The first variant looks at the dimension of sets with positive measure; the second variant examines the dimension of sets with full measure. For further discussion on dimension of measures, see [30].

Definition 2.5. For any finite Borel regular measure μ on $\mathbb{R}^n$, define the *lower* and *upper Hausdorff dimensions* of μ by

$$\underline{\dim}_H \mu = \inf\{\dim_H E : \mu(E) > 0\} \tag{2.5}$$

and

$$\overline{\dim}_H \mu = \inf\{\dim_H E : \mu(\mathbb{R}^n \setminus E) = 0\}. \tag{2.6}$$

Appendix A contains a bound on the lower Hausdorff dimension of harmonic measure on bounded domains in space.

2.2 Local Flatness and Reifenberg Flat Sets

Definition 2.6. Suppose that A and B are nonempty bounded subsets of a metric space X. The *Hausdorff distance* between A and B is

$$\mathrm{HD}(A, B) = \sup_{x \in A} \operatorname{dist}(x, B) + \sup_{y \in B} \operatorname{dist}(y, A). \tag{2.7}$$

Remark 2.7. If A, B are nonempty sets with closures $\overline{A}$, $\overline{B}$, then $\mathrm{HD}(A, B) = \mathrm{HD}(\overline{A}, \overline{B})$. The Hausdorff distance is a metric on the collection of nonempty bounded *closed* sets in X.

The Hausdorff distance may be used to quantify the "local flatness" of a set at a given location and scale. Here $G(n, n-1)$ denotes the collection of $(n-1)$-dimensional planes in $\mathbb{R}^n$ passing through the origin.

Definition 2.8. Let $A \subset \mathbb{R}^n$ be a nonempty set and let $x \in A$. For each $r > 0$, define

$$\theta_A(x, r) = \frac{1}{r} \min_{L \in G(n,n-1)} \mathrm{HD}(A \cap B(x, r), (x + L) \cap B(x, r)). \tag{2.8}$$

Remark 2.9. The minimum in (2.8) is achieved for some hyperplane $L \in G(n, n-1)$ by the compactness of $G(n, n-1)$. Note $\theta_A(x, r) \leq 2$ for all A, for all $x \in A$ and for all $r > 0$. If $A \subset \mathbb{R}^n$ is a nonempty set, then $\theta_A(x, r) = 0$ for every $x \in A$ and $r > 0$ if and only if the closure $\overline{A}$ is an $(n-1)$-plane.

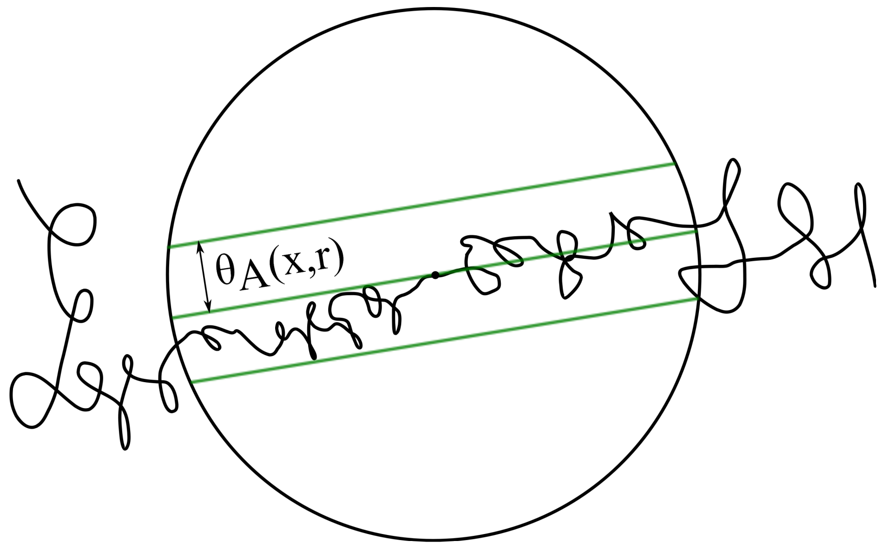

Figure 2.2: Local Flatness $\theta_A(x,r)$ of a Set A

Lemma 2.10. *Let $A \subset \mathbb{R}^n$ be a nonempty set and let $x, y \in A$. If $B(y, sr) \subset B(x, r)$, then $\theta_A(y, sr) \leq 6\theta_A(x, r)/s$.*

Proof. Applying a harmless translation, dilation and rotation, we may assume without loss of generality that $x = 0$, $r = 1$ and

$$\delta = \theta_A(0, 1) = \mathrm{HD}(A \cap B_1, L_0 \cap B_1) \tag{2.9}$$

where $L_0 = \{x \in \mathbb{R}^n : x_n = 0\}$. Fix $y \in A$ and $s > 0$ such that $B(y, s) \subset B_1$. To estimate $\theta_A(y, s)$ from above we will bound the Hausdorff distance between the set $A \cap B(y, s)$ and the plane $L_y \cap B(y, s)$ inside $B(y, s)$ where the $L_y = \{x \in \mathbb{R}^n : x_n = y_n\}$.

Suppose that $z \in A \cap B(y, s)$. Since $z \in A \cap B_1$, $\mathrm{dist}(z, L_0 \cap B_1) = |z - \pi(z)| \leq \delta$ where $\pi : \mathbb{R}^n \to L_0$ denotes the orthogonal projection onto L_0. Hence

$$\mathrm{dist}(z, L_y \cap B(y, s)) \leq \delta + \mathrm{dist}(\pi(z), L_y \cap B(y, s)). \tag{2.10}$$

To continue, note that since $\pi(z) \in \pi(B(y, s))$, $\mathrm{dist}(\pi(z), L_y \cap B(y, s)) = |y_n| \leq \delta$. Thus $\mathrm{dist}(z, L_y \cap B(y, s)) \leq 2\delta$ for all $z \in A \cap B(y, s)$.

Next suppose that $w \in L_y \cap B(y,s)$. Since $\pi(w) \in L_0 \cap B_1$, $\operatorname{dist}(\pi(w), A \cap B_1) \le \delta$. Hence $\operatorname{dist}(w, A \cap B_1) \le |w - \pi(w)| + \operatorname{dist}(\pi(w), A \cap B_1) \le |y_n| + \delta \le 2\delta$ for every $w \in L_y \cap B(y,s)$. But we really want to estimate $\operatorname{dist}(w, A \cap B(y,s))$. To that end choose $w' \in L_y \cap B(y, s-2\delta)$ such that $|w - w'| \le 2\delta$. From above we know $\operatorname{dist}(w', A \cap B_1) \le 2\delta$, say $\operatorname{dist}(w', A \cap B_1) = |w' - x'| \le 2\delta$ for some $x' \in A \cap B(0,1)$. Because $w' \in B(y, s-2\delta)$, it follows that $|x' - y| \le |x' - w'| + |w' - y| \le 2\delta + s - 2\delta \le s$ and $x' \in A \cap B(y,s)$. Thus $\operatorname{dist}(w', A \cap B(y,s)) \le 2\delta$. We conclude

$$\operatorname{dist}(w, A \cap B(y,s)) \le |w - w'| + \operatorname{dist}(w', A \cap B(y,s) \le 4\delta. \tag{2.11}$$

Therefore,

$$\theta_A(y,s) \le \frac{1}{s} \operatorname{HD}(A \cap B(y,s), L_y \cap B(y,s)) \le \frac{6\delta}{s} \tag{2.12}$$

as desired. In fact, we have only established (2.12) provided that $L_y \cap B(y, s-2\delta) \neq \emptyset$, i.e. when $s > 2\delta$. On the other hand, if $s \le 2\delta$, then $\theta_A(y,s) \le 2 < 3 \le 6\delta/s$, as well. □

Sets which are uniformly flat at all locations and scales first appeared in Reifenberg's solution of the Plateau problem [35].

Definition 2.11. Let $\delta > 0$. A nonempty set A is called *δ-Reifenberg flat* if there exists $r_0 > 0$ such that $\theta_A(x,r) \le \delta$ for all $x \in A$ and for all $0 < r < r_0$.

Definition 2.12. Let $\delta > 0$. A nonempty set A is called *locally δ-Reifenberg flat* if for every compact set $K \subset A$ there exists $r_0 > 0$ such that $\theta_A(x,r) \le \delta$ for all $x \in K$ and for all $0 < r < r_0$.

Definition 2.13. If A is δ-Reifenberg flat for every $\delta > 0$, we say A is *Reifenberg vanishing* or *Reifenberg flat with vanishing constant*.

Definition 2.14. If A is locally δ-Reifenberg flat for every $\delta > 0$, then we say A is *locally Reifenberg vanishing* or *locally Reifenberg flat with vanishing constant*.

Uniform flatness controls the Hausdorff dimension of a set.

Theorem 2.15 (Mattila and Vuorinen [31])**.** *If $A \subset \mathbb{R}^n$ is a (locally) δ-Reifenberg flat set, then* $\dim_H A \le n - 1 + C\delta^2$ *for some* $C = C(n) > 0$.

Corollary 2.16. *If $A \subset \mathbb{R}^n$ is (locally) Reifenberg vanishing, then* $\dim_H A = n - 1$.

For a complementary statement about Hausdorff dimension of "uniformly non-flat" sets, see Bishop and Jones [4] (for $n = 2$) and David [7] (for $n \ge 3$).

2.3 Blow-ups of Sets

Next we use Hausdorff distance to formalize the notion of "zooming in" on a closed set. Blow-ups commonly appear in the study of free boundary problems.

Definition 2.17. Let $A \subset \mathbb{R}^n$ be a nonempty closed set. A *blow-up B of A centered at* $x \in A$ is a closed set $B \subset \mathbb{R}^n$ such that for some sequence $r_i \downarrow 0$,

$$\lim_{i\to\infty} \mathrm{HD}\left(\frac{A - x}{r_i} \cap B_s, B \cap B_s\right) = 0 \qquad \text{for all } s > 0. \tag{2.13}$$

The existence of blow-ups is guaranteed by the following lemma.

Lemma 2.18 (Blaschke's selection theorem)**.** *Let $K \subset \mathbb{R}^n$ be a compact set. If $(A_k)_{k=1}^\infty$ is a sequence of nonempty closed subsets of K, then there exists a nonempty closed set $A \subset K$ and a subsequence $(A_{k_j})_{j=1}^\infty$ of $(A_k)_{k=1}^\infty$ such that* $\mathrm{HD}(A_{k_j}, A) \to 0$ *as* $j \to \infty$.

Proof. For example, see Rogers [36], page 91. □

Remark 2.19. Let $A \subset \mathbb{R}^n$ be a nonempty closed set and let $x \in A$. By Blaschke's selection theorem, for every sequence $r_i \downarrow 0$ the set A admits a blow-up centered at x along some subsequence of r_i. Thus blow-ups of A exist at every $x \in A$.

Example 2.20. The plane $\mathbb{R}^{n-1} \times \{0\}$ is the unique blow-up of the unit sphere S^{n-1} at the north pole $N = (\bar{0}, 1)$.

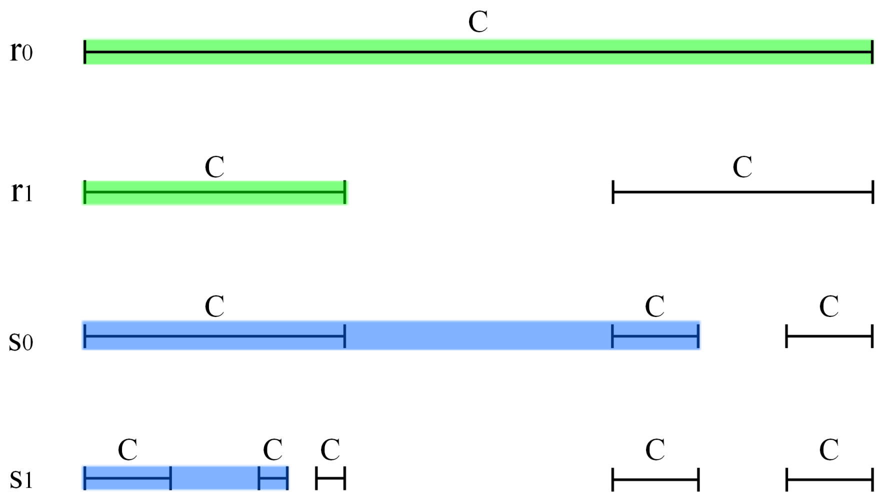

Figure 2.3: Blow-ups of the Cantor Middle-Thirds Set

Example 2.21. Let $C \subset [0,1]$ denote the Cantor Middle-Thirds Set, i.e. $C = \bigcap_{i\geq 0} C_i$ where $C_0 = [0,1]$ and C_{i+1} is the closed set obtained by removing the "middle-third" of each connected component (interval) of C_i. The Cantor Middle-Thirds Set has a rich tangent structure, which we now illustrate. By self-similarity,

$$C = \frac{1}{3}C \cup \left(\frac{2}{3} + \frac{1}{3}C\right) = \frac{1}{3}C \cup \left(\frac{6}{9} + \frac{1}{9}C\right) \cup \left(\frac{8}{9} + \frac{1}{9}C\right). \tag{2.14}$$

Let $x = 0$, let $r_i = 3^{-i}$ for each $i \geq 0$ and let $s_i = (7/9)r_i$ for all $i \geq 0$. Then C has distinct blow-ups centered at 0 as $i \to \infty$ along the sequences r_i and s_i (see Figure 2.3). To verify this assertion, simply note that, by (2.14),

$$\frac{C}{r_i} \cap B_1 = C \quad \text{for all } i \geq 0, \tag{2.15}$$

while

$$\frac{C}{s_i} \cap B_1 = \frac{9}{7}\left[\frac{1}{3}C \cup \left(\frac{6}{9} + \frac{1}{9}C\right)\right] \neq C \quad \text{for all } i \geq 0. \tag{2.16}$$

Thus, the blow-ups of a closed set A centered at $x \in A$ are not always unique.

2.4 Weak Convergence of Measures

A *Radon measure* μ on $\mathbb{R}^n$ is a positive Borel regular outer measure on $\mathbb{R}^n$ that is finite on compact sets; the *support* $\operatorname{spt}\mu$ of μ is the smallest closed set E such that $\mu(\mathbb{R}^n \setminus E) = 0$. A sequence $(\mu_i)_{i=1}^{\infty}$ of Radon measures on $\mathbb{R}^n$ *converges weakly* to a Radon measure μ and we write $\mu_i \rightharpoonup \mu$ provided

$$\lim_{i\to\infty} \int f d\mu_i = \int f d\mu \quad \text{for all } f \in C_c(\mathbb{R}^n). \tag{2.17}$$

Of course, to test for weak convergence one only needs to check that (2.17) holds on a class of functions smaller than $C_c(\mathbb{R}^n)$. For example, either the class $C_c^{\infty}(\mathbb{R}^n)$ of smooth functions with compact support or the class $\operatorname{Lip}_c(\mathbb{R}^n)$ of Lipschitz functions with compact support suffice.

Below we use a quantitative version of weak convergence (introduced by Preiss [34]), which captures the idea that $\mu_i \rightharpoonup \mu$ exactly when μ_i "gets close to" μ on the ball B_r for every $r > 0$.

Definition 2.22. Let μ and ν be Radon measure on $\mathbb{R}^n$. For all $r > 0$ define

$$F_r(\mu,\nu) = \sup\left\{ \left| \int f d\mu - \int f d\nu \right| : f \geq 0, \operatorname{Lip} f \leq 1, \operatorname{spt} f \subset B_r \right\} \tag{2.18}$$

where $\operatorname{Lip} f$ and $\operatorname{spt} f$ denote the Lipschitz constant and the support of a function f.

As an easy exercise one checks F_r is a semi-metric on the set of Radon measures on $\mathbb{R}^n$ and a metric on the subset of measures supported in B_r. If $r \leq s$ then $F_r(\mu,\nu) \leq F_s(\mu,\nu)$. If μ is a Radon measure on $\mathbb{R}^n$, then

$$F_r(\mu) := F_r(\mu, 0) = \int_0^r \mu(B_s) ds \tag{2.19}$$

Indeed

$$F_r(\mu,0) = \sup\left\{ \int f d\mu : f \geq 0, \operatorname{Lip} f \leq 1, \operatorname{spt} f \subset B_r \right\} = \int \operatorname{dist}(z, \mathbb{R}^n \setminus B_r) d\mu(z)$$
$$= \int_0^r \mu\{z : \operatorname{dist}(z, \mathbb{R}^n \setminus B_r) > s\} ds = \int_0^r \mu(B_{r-s}) ds = \int_0^r \mu(B_s) ds.$$

Observe $F_r(\mu) < \infty$ for any Radon measure μ, since μ is finite on compact sets. In fact,

$$\frac{r}{2}\mu(B_{r/2}) \leq F_r(\mu) \leq r\mu(B_r) \quad \text{for all } r > 0. \tag{2.20}$$

We now state the relationship between weak convergence of Radon measures and F_r.

Lemma 2.23 ([29] Lemma 14.13)**.** *Suppose that $\mu, \mu_1, \mu_2, \dots$ are Radon measures on $\mathbb{R}^n$. Then $\mu_i \rightharpoonup \mu$ if and only if $\lim_{i\to\infty} F_r(\mu_i, \mu) = 0$ for all $r > 0$.*

Proposition 2.24 ([34] Proposition 1.12)**.** *The Radon measures on $\mathbb{R}^n$ admit a complete separable metric*

$$\sum_{i=1}^{\infty} 2^{-i} \min(1, F_i(\mu, \nu)) \tag{2.21}$$

whose topology is equivalent to the topology of weak convergence of Radon measures.

Remark 2.25. The family of semi-metrics F_r is related to a distance between probability measures in a compact metric space, which is known by various names in the literature. For any compact metric space X, the *Kantorovich-Rubinstein* formula

$$\sup\left\{\int_X f d(\mu - \nu) : \operatorname{Lip} f \leq 1\right\} \tag{2.22}$$

defines a complete separable metric on the collection of probability measures on X whose topology is equivalent to the weak convergence of probability measures [17]. For further discussion, we refer the reader to the bibliographical notes in Chapter 6 of [38].

2.5 Tangent Measures and Cones of Measures

Tangent measures (introduced by Preiss [34]) are a measure-theoretic analogue of blow-ups of a closed set. Given $x \in \mathbb{R}^n$ and $r > 0$, write $T_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ for translation by x followed by dilation by r,

$$T_{x,r}(y) = \frac{y - x}{r} \quad \text{for all } y \in \mathbb{R}^n. \tag{2.23}$$

The image measure $T_{x,r}[\mu]$ of a Radon measure μ, which acts on a set $E \subset \mathbb{R}^n$ by

$$T_{x,r}[\mu](E) = \mu(T_{x,r}^{-1}(E)) = \mu(x + rE), \tag{2.24}$$

is also Radon since $T_{x,r}$ is a homeomorphism. When $E = B_1$, we interpret (2.24) as saying $T_{x,r}[\mu]$ "blows up" $B(x,r)$ to the unit ball B_1, in the sense that $\mu(B(x,r)) = T_{x,r}[\mu](B_1)$. Integration against $T_{x,r}[\mu]$ obeys

$$\int f(z) dT_{x,r}[\mu](z) = \int f\left(\frac{z-x}{r}\right) d\mu(z) \tag{2.25}$$

whenever at least one of the integrals is defined. Furthermore, the map $T_{x,r}$ interacts with the semi-metric F_r as follows. For all Radon measures μ, ν on $\mathbb{R}^n$ and all radii $r, s > 0$:

$$T_{x,rs} = T_{0,s} \circ T_{x,r}, \tag{2.26}$$

$$T_{x,rs}[\mu] = T_{0,s}[T_{x,r}[\mu]], \tag{2.27}$$

$$F_{rs}(\mu) = sF_r(T_{0,s}[\mu]), \tag{2.28}$$

$$F_{rs}(\mu, \nu) = sF_r(T_{0,s}[\mu], T_{0,s}[\nu]). \tag{2.29}$$

We are ready to present Preiss' definition [34] of a tangent measure to a Radon measure. The basic idea is to take a sequence of blow-ups $T_{x,r_i}[\mu]$ of the measure μ with scales $r_i > 0$ shrinking to zero and then normalize by some constants $c_i > 0$ so that the limit converges.

Definition 2.26. Let μ be a non-zero Radon measure and let $x \in \operatorname{spt} \mu$. A non-zero Radon measure ν is a *tangent measure* of μ at x and we write $\nu \in \operatorname{Tan}(\mu, x)$ if there exists sequences $r_i \downarrow 0$ and $c_i > 0$ such that

$$c_i T_{x,r_i}[\mu] \rightharpoonup \nu. \tag{2.30}$$

In [22], Kenig and Toro introduced a variant of tangent measures where the base point $x \in \operatorname{spt} \mu$ in (2.30) is replaced with a sequence of points $x_i \to x \in \operatorname{spt} \mu$.

Definition 2.27. Let μ be a non-zero Radon measure and let $x \in \operatorname{spt} \mu$. A non-zero Radon measure ν is said to be a *pseudotangent measure* of μ at x and we write $\nu \in \Psi\text{–}\operatorname{Tan}(\mu, x)$ if there exists sequences $x_i \to x$ (with $x_i \in \operatorname{spt} \mu$), $r_i \downarrow 0$ and $c_i > 0$ such that

$$c_i T_{x_i,r_i}[\mu] \rightharpoonup \nu \tag{2.31}$$

Clearly every tangent measure is a pseudotangent measure: $\operatorname{Tan}(\mu, x) \subset \Psi\text{–}\operatorname{Tan}(\mu, x)$ for every Radon measure μ and $x \in \operatorname{spt}\mu$. Also the collection of tangent measures at a point is non-empty under mild assumptions on the measure. For example, if $x \in \operatorname{spt}\mu$ and one of the conditions

- $\overline{D}^s(\mu, x) = \limsup_{r\downarrow 0} \mu(B(x,r))/r^s \in (0,\infty)$ for some $0 < s < \infty$, or

- $\limsup_{r\downarrow 0} \mu(B(x,2r))/\mu(B(x,r)) < \infty$,

hold, then $\operatorname{Tan}(\mu, x) \neq \emptyset$ by compactness of Radon measures in the weak topology.

Taking blow-ups of a measure at a point is a closed operation in the sense that tangent measures to tangent measures are tangent measures.

Lemma 2.28 ([3] Lemma 2.6)**.** *Let μ be a non-zero Radon measure on $\mathbb{R}^n$ and let $x \in \operatorname{spt}\mu$. If $\nu \in \operatorname{Tan}(\mu, x)$, then $\operatorname{Tan}(\nu, 0) \subset \operatorname{Tan}(\mu, x)$.*

Proof. Fix $\rho \in \operatorname{Tan}(\nu, 0)$. Let $r_i, s_i \downarrow 0$ and $c_i, d_i > 0$ be sequences such that $c_i T_{x,r_i}[\mu] \rightharpoonup \nu$ and $d_i T_{0,s_i}[\nu] \rightharpoonup \rho$. Since $c_i T_{x,r_i}[\mu] \rightharpoonup \nu$, we know that $\lim_{i\to\infty} F_1(c_i T_{x,r_i}[\mu], \nu) = 0$ by Lemma 2.23. Choose a subsequence $(c_{i(j)}, r_{i(j)})$ of (c_i, r_i) such that

$$F_1(c_{i(j)} T_{x,r_{i(j)}}[\mu], \nu) \leq \frac{1}{j}\left(\frac{s_j}{d_j}\right) \quad \text{for all } j \geq 1. \tag{2.32}$$

After relabeling $(c_{i(j)}, r_{i(j)})$, we may assume that

$$F_1(c_j T_{x,r_j}[\mu], \nu) \leq \frac{1}{j}\left(\frac{s_j}{d_j}\right) \quad \text{for all } j \geq 1. \tag{2.33}$$

Fix $r > 0$. Since F_r is a semi-metric,

$$F_r(c_j d_j T_{x,r_j s_j}[\mu], \rho) \leq F_r(c_j d_j T_{x,r_j s_j}[\mu], d_j T_{0,s_j}[\nu]) + F_r(d_j T_{0,s_j}[\nu], \rho). \tag{2.34}$$

On one hand, we get $\lim_{j\to\infty} F_r(d_j T_{0,s_j}[\nu], \rho) = 0$ since $d_j T_{0,s_j}[\nu] \rightharpoonup \rho$. On the other hand, for all j sufficiently large such that $s_j r \leq 1$,

$$\begin{aligned} F_r(c_j d_j T_{x,r_j s_j}[\mu], d_j T_{0,s_j}[\nu]) &= d_j F_r(T_{0,s_j}[c_j T_{x,r_j}[\mu]], T_{0,s_j}[\nu]) \\ &= \frac{d_j}{s_j} F_{s_j r}(c_j T_{x,r_j}[\mu], \nu) \\ &\leq \frac{d_j}{s_j} F_1(c_j T_{x,r_j}[\mu], \nu) \leq \frac{1}{j}. \end{aligned} \tag{2.35}$$

Thus, $\lim_{j\to\infty} F_r(c_j d_j T_{x,r_j s_j}[\mu],\rho) = 0$ for all $r > 0$. By Lemma 2.23, we conclude that $c_j d_j T_{x,r_j s_j}[\mu] \rightharpoonup \rho$ and $\rho \in \operatorname{Tan}(\mu, x)$. Therefore, $\operatorname{Tan}(\nu, 0) \subset \operatorname{Tan}(\mu, x)$. □

At almost every point in the support of a measure, the collection of tangent measures at a point is closed under translation.

Theorem 2.29 ([29] Theorem 14.16)**.** *Let μ be a non-zero Radon measure. At μ-almost every $x \in \operatorname{spt}\mu$ the following holds: if $\nu \in \operatorname{Tan}(\mu, x)$ and $y \in \operatorname{spt}\nu$, then*

1. $T_{y,1}[\nu] \in \operatorname{Tan}(\mu, x)$,

2. $\operatorname{Tan}(\nu, y) \subset \operatorname{Tan}(\mu, x)$.

Proof Sketch. The proof of (1) uses the separability of Radon measures in the topology generated by the semi-metrics F_r. Statement (2) follows quickly from (1), the composition law $T_{y,r_i}[\nu] = T_{0,r_i}[T_{y,1}[\nu]]$ and Lemma 2.28. □

Preiss [34] also introduced cones of measures, or collections of positive measures which are invariant under scaling.

Definition 2.30. A nonempty collection $\mathcal{M}$ of non-zero Radon measures on $\mathbb{R}^n$ is a *cone* provided $c\psi \in \mathcal{M}$ whenever $\psi \in \mathcal{M}$ and $c > 0$.

Definition 2.31. Let $\mathcal{M}$ be a cone of non-zero Radon measures. We say $\mathcal{M}$ is

- *dilation invariant* if $T_{0,r}[\psi] \in \mathcal{M}$ for all $\psi \in \mathcal{M}$ and $x \in \operatorname{spt}\psi$,
- *translation invariant* if $T_{x,1}[\psi] \in \mathcal{M}$ for all $\psi \in \mathcal{M}$ and $x \in \operatorname{spt}\psi$.

If $\mathcal{M}$ is a dilation invariant cone of Radon measures, then for all $r > 0$ there is $\mu \in \mathcal{M}$ such that $F_r(\mu) > 0$. Indeed choose any measure $\psi \in \mathcal{M}$. Then $F_s(\psi) > 0$ for some $s > 0$ because $\psi \neq 0$. For any $r > 0$,

$$F_r(T_{0,s/r}[\psi]) = \frac{r}{s} F_{r(s/r)}(\psi) = \frac{r}{s} F_s(\psi) > 0. \tag{2.36}$$

Since $\mathcal{M}$ is closed under dilations, $\psi_r = T_{0,s/r}[\psi] \in \mathcal{M}$ satisfies $F_r(\psi_r) > 0$. In particular, since $F_1(\psi_1) > 0$ and $\mathcal{M}$ is closed under scaling the following set is non-empty.

Definition 2.32. The *basis* of a dilation invariant $\mathcal{M}$ is the subset $\{\psi \in \mathcal{M} : F_1(\psi) = 1\}$.

Lemma 2.33 ([34] Remark 2.1)**.** *Let $\mathcal{M}$ be a dilation invariant cone. In the topology of weak convergence of Radon measures, $\mathcal{M}$ is relatively closed (relatively compact) in the collection of all non-zero Radon measures if and only if the basis of $\mathcal{M}$ is closed (compact).*

We are already familiar with the canonical example of a dilation invariant cone.

Lemma 2.34 ([34] Remark 2.3)**.** *If $\mathrm{Tan}(\mu, x) \neq \emptyset$, then $\mathrm{Tan}(\mu, x)$ is a dilation invariant cone with a closed basis.*

Following [34] we define a normalized version of F_r for the distance of a measure σ to a dilation invariant cone $\mathcal{M}$ of measures as follows:

$$d_r(\sigma, \mathcal{M}) = \inf \left\{ F_r\left(\frac{\sigma}{F_r(\sigma)}, \psi\right) : \psi \in \mathcal{M} \text{ and } F_r(\psi) = 1 \right\}. \tag{2.37}$$

If $F_r(\sigma) = 0$ we set $d_r(\sigma, \mathcal{M}) = 1$.

Under certain conditions, the cone of tangent measures of a given measure at a point is connected in topology of weak convergence of Radon measures, in the following sense.

Theorem 2.35 ([20] Corollary 2.1)**.** *Let $\mathcal{F}$ and $\mathcal{M}$ be dilation invariant cones, $\mathcal{F} \subset \mathcal{M}$. Assume that*

1. *Both $\mathcal{F}$ and $\mathcal{M}$ have compact bases,*

2. *There exists $\epsilon_0 > 0$ such that $\psi \in \mathcal{M}$ and $d_r(\psi, \mathcal{F}) < \epsilon_0$ for all $r \geq r_0$ implies $\psi \in \mathcal{F}$.*

If $\mathrm{Tan}(\mu, x) \subset \mathcal{M}$ and $\mathrm{Tan}(\mu, x) \cap \mathcal{F} \neq \emptyset$, then $\mathrm{Tan}(\mu, x) \subset \mathcal{F}$.

Remark 2.36. In Chapter 5 we provide an example which demonstrates Theorem 2.35 is false if one replaces tangent measures $\mathrm{Tan}(\mu, x)$ with pseudotangent measures $\Psi\text{–}\mathrm{Tan}(\mu, x)$. See Example 5.28.

We end this section with two conditions that guarantee a dilation invariant cone has a compact basis. Additional criterion may be found in [34].

Proposition 2.37 ([34] Proposition 2.2)**.** *Assume $\mathcal{M}$ is a dilation invariant cone with a closed basis. Then $\mathcal{M}$ has a compact basis if and only if there exists a finite number $q \geq 1$ such that $\psi(B(0,2r)) \leq q\psi(B(0,r))$ for all $\psi \in \mathcal{M}$ and $r > 0$.*

Corollary 2.38 ([34] Corollary 2.7)**.** *Let μ be a non-zero Radon measure. If $x \in \operatorname{spt}\mu$ and $\limsup_{r\downarrow 0} \mu(B(x,2r))/\mu(B(x,r)) < \infty$ then $\operatorname{Tan}(\mu, x)$ has a compact basis.*

Chapter 3

ZERO SETS OF HARMONIC POLYNOMIALS

The focus of this chapter is on harmonic polynomials $h : \mathbb{R}^n \to \mathbb{R}$ and their zero sets

$$\Sigma_h = \{x \in \mathbb{R}^n : h(x) = 0\}. \quad (3.1)$$

When h is a polynomial of degree one, h is automatically harmonic and its zero set Σ_h of h is an affine $(n-1)$-plane. What can be said about higher degree harmonic polynomials? First Liouville's theorem (a harmonic function on $\mathbb{R}^n$ bounded above or below is constant) guarantees that the zero set Σ_h is nonempty for every nonconstant harmonic polynomial. Because polynomials are continuous functions, we know that Σ_h is a closed set. Moreover, by the mean value property, Σ_h has no isolated points; and by the maximum principle, each connected component of $\mathbb{R}^n \setminus \Sigma_h$ is unbounded.

The number of connected components of $\mathbb{R}^n \setminus \Sigma_h$ (call it N_h) depends on the dimension. In the plane $(n = 2)$, $N_h = 2d$ whenever h is a harmonic polynomial and $d = \deg h \geq 1$. Higher dimensions $(n \geq 3)$ display a greater variety of behaviors. The most is known when $n = 3$ and h is homogeneous. If $h : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous harmonic polynomial of degree $d \geq 1$, then $N_h \leq d^2 + 1$ by the Courant nodal domain theorem. For this and improved upper bounds, see Leydold [28] where the sharp upper bound on N_h is computed for $d \leq 6$. Conversely, Lewy [27] established the lower bounds $N_h \geq 2$ when $d \geq 1$ is odd, $N_h \geq 3$ when $d \geq 2$ is even, and these are the best possible (for examples showing these bounds can be obtained when $d = 2$ and $d = 3$, see Figure 3.1 and 3.3.) On the other hand, the polynomial $h(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$ shows the zero set of a homogeneous harmonic polynomial of even degree can divide space into two components in dimensions $n \geq 4$. (Hint: To check this, a motivated reader can find a piecewise linear path from any $x \in \mathbb{R}^4$ such that $x_1^2 + x_2^2 - x_3^2 - x_4^2 > 0$ to $(1, 0, 0, 0)$.)

Zero Sets Dividing Space into Three or More Components

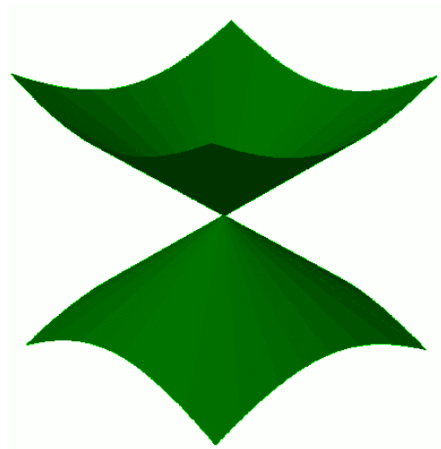

Figure 3.1: Zero Set of $2z^2 - x^2 - y^2$

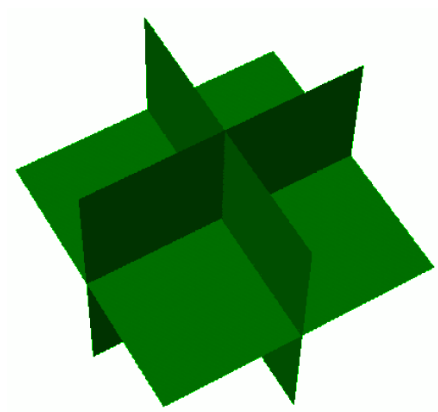

Figure 3.2: Zero Set of xyz

Zero Sets Dividing Space into Two Components

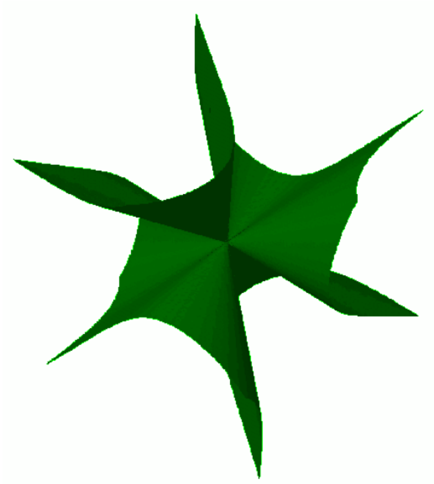

Figure 3.3: Zero Set of $x^2(y-z)+y^2(z-x)+z^2(x-y)-xyz$

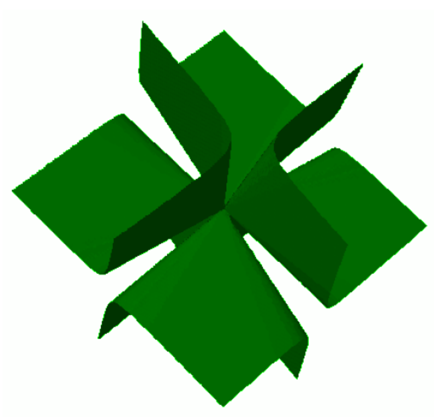

Figure 3.4: Zero Set of $x^2(y-z)+y^2(z-x)+z^2(x-y)-10xyz$

Table 3.1: Sharp Bounds on the Number of Connected Components of $\mathbb{R}^n \setminus \Sigma_h$

$n = 2$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
Exact Value	2	4	6	8	10	12	14

$n = 3$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$	$d = 7$
Upper Bound	2	4	8	12	18	24	?
Lower Bound	2	3	2	3	2	3	2

The regularity of zero sets of solutions of elliptic equations with smooth coefficients has been studied by Hardt and Simon [12]. Applied to the zero sets of harmonic polynomials, their result can be stated as follows:

Theorem 3.1 (Hardt and Simon [12])**.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial of degree $d \geq 2$. The zero set Σ_h of h decomposes into a disjoint union*

$$\Sigma_h = R_h \cup S_h \tag{3.2}$$

where the "regular set" $R_h = \{x \in \Sigma_h : Dh(x) \neq 0\}$ is an embedded $(n-1)$-dimensional C^1 submanifold of $\mathbb{R}^n$ and the "singular set" $S_h = \{x \in \Sigma_h : Dh(x) = 0\}$ is closed and contained in a countable union of embedded $(n-2)$-dimensional C^1 submanifolds of $\mathbb{R}^n$.

Our main goal in this chapter is to prove that the decomposition of the zero set Σ_h into a regular set R_h and a singular set S_h can be characterized in terms of "local flatness", uniformly across all harmonic polynomials of degree d. Roughly put, we demonstrate that at each point in Σ_h either the zero set becomes arbitrarily flat on small scales or the zero set stays far away from a plane at every scale. The two alternatives correspond exactly with the sets R_h and S_h, respectively. Recall that the quantity $\theta_{\Sigma_h}(x, r)$ (see Definition 2.8) measures how closely one can approximate $\Sigma_h \cap B(x, r)$ by an $(n-1)$-plane through the point $x \in \Sigma_h$ in the Hausdorff distance.

Theorem 3.2. *For all $n \geq 2$ and $d \geq 2$ there exists a constant $\delta_{n,d} > 0$ such that for any harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree d and for any $x \in \Sigma_h = \{y \in \mathbb{R}^n : h(y) = 0\}$,*

$$Dh(x) = 0 \quad \Longleftrightarrow \quad \theta_{\Sigma_h}(x,r) \geq \delta_{n,d} \quad \text{for all } r > 0, \tag{3.3}$$

$$Dh(x) \neq 0 \quad \Longleftrightarrow \quad \theta_{\Sigma_h}(x,r) < \delta_{n,d} \quad \text{for some } r > 0. \tag{3.4}$$

Moreover, if $\theta_{\Sigma_h}(x,r) < \delta_{n,d}$, then $\theta_{\Sigma_h}(x,sr) < s\sqrt{5}(d-1)\delta_{n,d}^{-1}$ for all $s \in (0,1)$.

Remark 3.3. The assertion of Theorem 3.2 that at a singularity the zero set Σ_h is not flat is a special feature of harmonic polynomials which does not hold for general polynomials. For example, $p(x,y) = x^4 + y^4 - y^2$ has a singularity at the origin (i.e. $Dp(0,0) = 0$); however, its zero set is locally flat near the origin with vanishing constant. See Figure 3.5. In fact, the unique blow-up of Σ_p at the origin is the x-axis.

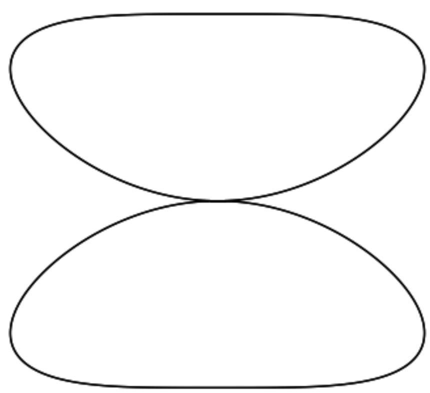

Figure 3.5: Zero Set of $x^4 + y^4 - y^2$

To prove Theorem 3.2 we identify a certain quantity $\zeta_1(h,x,r) \in [0,\infty]$, which depends continuously on the coefficients of h. This quantity identifies whether or not $Dh(x)$ vanishes and controls $\theta_{\Sigma_h}(x,r)$ from above. Moreover, at any $x \in \Sigma_h$ the number $\zeta_1(h,x,r)$ decays linearly in the variable r in the sense that $\zeta_1(h,x,sr) \leq s\zeta_1(h,x,r)$ for all $s \in (0,1)$. The critical step in the proof of the theorem is to show that for harmonic polynomials $\zeta_1(h,x,r) < \infty$ whenever $\theta_{\Sigma_h}(x,r)$ is sufficiently small (Proposition 3.30).

The organization of the rest of the chapter is as follows. In §3.1 we define the relative size $\zeta_k(p,x,r)$ of the homogeneous part of degree k of a polynomial p on the ball $B(x,r)$, and record the basic properties of these numbers. Section 3.2 contains auxiliary estimates for spherical harmonics (harmonic polynomials restricted to the unit sphere), which we need in below in §3.4 and in Chapter 4. In §3.3 we discuss some aspects of convergence of zero sets associated to a sequence of polynomials. We use these results and a blow-up type argument to finish the proof of Theorem 3.2 in §3.4.

3.1 Relative Size of Homogeneous Parts

Let $x \in \mathbb{R}^n$. A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $d \geq 1$ decomposes as

$$p(z) = p_d^{(x)}(z-x) + \cdots + p_1^{(x)}(z-x) + p_0^{(x)}(z-x) \tag{3.5}$$

where each non-zero term $p_k^{(x)}$ is a *homogenous* polynomial of degree k, i.e.

$$p_k^{(x)}(ty) = t^k p_k^{(x)}(y) \quad \text{for all } t \in \mathbb{R} \text{ and } y \in \mathbb{R}^n. \tag{3.6}$$

We call $p_k^{(x)}$ the *homogeneous part* of p of *degree* k with *center* x. By Taylor's theorem,

$$p_k^{(x)}(y) = \sum_{|\alpha|=k} \frac{D^\alpha p(x)}{\alpha!} y^\alpha \quad \text{for all } y \in \mathbb{R}^n. \tag{3.7}$$

In the sequel, it will be convenient to quantify the relative sizes of homogeneous parts.

Definition 3.4. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \geq 1$. For every $0 \leq k \leq d$, $x \in \mathbb{R}^n$ and $r > 0$, define

$$\zeta_k(p,x,r) = \max_{j \neq k} \frac{\|p_j^{(x)}\|_{L^\infty(B_r)}}{\|p_k^{(x)}\|_{L^\infty(B_r)}} \in [0,\infty]. \tag{3.8}$$

Remark 3.5. Definition 3.4 generalizes the two quantities $\zeta(h)$ and $\zeta_*(h)$ associated to a harmonic polynomial h, which appeared in Badger [3] (see Lemma 4.3 and Lemma 4.5). In the present notation, if $h = h_d^{(0)} + \cdots + h_j^{(0)}$ is a harmonic polynomial of degree $d \geq 1$ such that $h(0) = 0$ and $h_j^{(0)} \neq 0$, then $\zeta(h) = \zeta_d(h,0,1)$ and $\zeta_*(h) = \zeta_j(h,0,1)$.

Because $\zeta_k(p, x, r)$ measures the *relative* size of homogeneous parts of a polynomial, scaling p does not affect ζ_k. This simple observation will enable proofs via normal families (for example, see the proof of Proposition 3.30), by allowing us to assume a sequence of polynomials with certain properties has uniformly bounded coefficients.

Lemma 3.6. *If $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$ and $c \in \mathbb{R} \setminus \{0\}$, then $\zeta_k(cp, x, r) = \zeta_k(p, x, r)$ for all $0 \leq k \leq d$, $x \in \mathbb{R}^n$ and $r > 0$.*

Proof. Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$, and let $c \in \mathbb{R} \setminus \{0\}$. Since $(cp)_k^{(x)} = c(p_k^{(x)})$ for all $0 \leq k \leq d$,

$$\begin{aligned} \zeta_k(cp, x, r) = \max_{j \neq k} \frac{\|cp_j^{(x)}\|_{L^\infty(B_r)}}{\|cp_k^{(x)}\|_{L^\infty(B_r)}} &= \max_{j \neq k} \frac{|c| \cdot \|p_j^{(x)}\|_{L^\infty(B_r)}}{|c| \cdot \|p_k^{(x)}\|_{L^\infty(B_r)}} \\ &= \max_{j \neq k} \frac{\|p_j^{(x)}\|_{L^\infty(B_r)}}{\|p_k^{(x)}\|_{L^\infty(B_r)}} = \zeta_k(p, x, r) \end{aligned} \tag{3.9}$$

for all $x \in \mathbb{R}^n$ and all $r > 0$. □

The quantity $\zeta_k(p, x, r)$ also behaves well under translation and dilation.

Lemma 3.7. *Suppose that $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$. If $z \in \mathbb{R}^n$, then $\zeta_k(p(\cdot + z), x, r) = \zeta_k(p, x + z, r)$ for all $x \in \mathbb{R}^n$, for all $r > 0$ and for all $0 \leq k \leq d$.*

Proof. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \geq 1$, fix $x \in \mathbb{R}^n$ and define $q : \mathbb{R}^n \to \mathbb{R}$ by $q(y) = p(y + z)$ for all $y \in \mathbb{R}^n$. Then q is a polynomial of degree d. Moreover, for all $0 \leq k \leq d$,

$$q_k^{(x)}(y) = \sum_{|\alpha|=k} \frac{D^\alpha q(x)}{\alpha!} y^\alpha = \sum_{|\alpha|=k} \frac{D^\alpha p(x+z)}{\alpha!} y^\alpha = p_k^{(x+z)}(y) \quad \text{for all } y \in \mathbb{R}^n. \tag{3.10}$$

Thus, $q_k^{(x)} = p_k^{(x+z)}$ for all $x \in \mathbb{R}^n$ and for all $0 \leq k \leq d$. It immediately follows that $\zeta_k(q, x, r) = \zeta_k(p, x + z, r)$ for all $0 \leq k \leq d$, for all $x \in \mathbb{R}^n$ and for all $r > 0$. □

Lemma 3.8. *If $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$ and $t > 0$, then $\zeta_k(p(t\cdot), x, r) = \zeta_k(p, tx, tr)$ for all $x \in \mathbb{R}^n$, for all $r > 0$ and for all $0 \leq k \leq d$.*

Proof. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree $d \geq 1$, fix $t > 0$ and define $q : \mathbb{R}^n \to \mathbb{R}$ by $q(y) = p(ty)$ for all $y \in \mathbb{R}^n$. Then q is a polynomial of degree d and for all $0 \leq k \leq d$,

$$q_k^{(x)}(y) = \sum_{|\alpha|=k} \frac{D^\alpha q(x)}{\alpha!} y^\alpha = t^k \sum_{|\alpha|=k} \frac{D^\alpha p(tx)}{\alpha!} y^\alpha = t^k p_k^{(tx)}(y) \quad \text{for all } y \in \mathbb{R}^n. \tag{3.11}$$

Hence $q_k^{(x)} = t^k p_k^{(tx)}$ for all $0 \leq k \leq d$. It follows that

$$\begin{aligned} \zeta_k(q, x, r) = \max_{j \neq k} \frac{\|q_j^{(x)}\|_{L^\infty(B_r)}}{\|q_k^{(x)}\|_{L^\infty(B_r)}} &= \max_{j \neq k} \frac{t^j \|p_j^{(tx)}\|_{L^\infty(B_r)}}{t^k \|p_k^{(tx)}\|_{L^\infty(B_r)}} \\ &= \max_{j \neq k} \frac{\|p_j^{(tx)}\|_{L^\infty(B_{tr})}}{\|p_k^{(tx)}\|_{L^\infty(B_{tr})}} = \zeta_k(p, tx, tr) \end{aligned} \tag{3.12}$$

for all $x \in \mathbb{R}^n$, for all $r > 0$ and for all $0 \leq k \leq d$. □

The magnitude of $\zeta_k(p, x, r)$ identifies homogeneous polynomials and the vanishing of homogeneous parts of polynomials. For example, $p(x) = 0$ if and only if $\zeta_0(p, x, r) = \infty$, and $Dp(x) = 0$ if and only if $\zeta_1(p, x, r) = \infty$.

Lemma 3.9. *If $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$, $x \in \mathbb{R}^n$ and $0 \leq k \leq d$, then*

1. $\zeta_k(p, x, r) = 0$ *for all* $r > 0$ *if and only if* $p_k^{(x)} = p(\cdot + x)$*;*
2. $\zeta_k(p, x, r) > 0$ *for all* $r > 0$ *if and only if* $p_k^{(x)} \neq p(\cdot + x)$*;*
3. $\zeta_k(p, x, r) < \infty$ *for all* $r > 0$ *if and only if* $p_k^{(x)} \neq 0$*; and,*
4. $\zeta_k(p, x, r) = \infty$ *for all* $r > 0$ *if and only if* $p_k^{(x)} = 0$*.*

Proof. We leave this exercise in the definition of $\zeta_k(p, x, r)$ to the reader. □

The value of $\zeta_k(p, x, r)$ depends continuously on the coefficients of the polynomial p. To make this statement precise, we first make a definition.

Definition 3.10. A sequence of polynomials $(p^i)_{i=1}^\infty$ in $\mathbb{R}^n$ converges *in coefficients* to a polynomial p in $\mathbb{R}^n$ if $d = \max_i \deg p^i < \infty$ and $D^\alpha p^i(0) \to D^\alpha p(0)$ for every $|\alpha| \leq d$.

Lemma 3.11. *For every $k \geq 0$, $\zeta_k(p, x, r)$ is jointly continuous in p, x and r. That is,*

$$\zeta_k(p^i, x_i, r_i) \to \zeta_k(p, x, r) \tag{3.13}$$

whenever $p^i \to p$ in coefficients, $x_i \to x \in \mathbb{R}^n$ and $r_i \to r \in (0, \infty)$.

Proof. Let $(p^i)_{i=1}^{\infty}$ be a sequence of polynomials in $\mathbb{R}^n$ such that $p^i \to p$ in coefficients to a nonconstant polynomial p and let $d = \max_i \deg p_i < \infty$. There are two cases. If $k > d$, then $p_k^{i(x_i)} = 0$ for all $i \geq 1$ and $p_k^{(x)} = 0$. Hence

$$\zeta_k(p^i, x_i, r_i) = \zeta_k(p, x, r) = \infty \quad \text{for all } i \geq 1, \text{ for all } k > d, \tag{3.14}$$

by Lemma 3.9. Otherwise $0 \leq k \leq d$. Since

$$D^\alpha p(x) = \sum_{|\beta| \leq d - |\alpha|} \frac{D^{\alpha+\beta} p(0)}{\beta!} x^\beta \quad \text{for all } x \in \mathbb{R}^n \text{ and } |\alpha| \leq d, \tag{3.15}$$

convergence in coefficients implies that $D^\alpha p^i(x) \to D^\alpha p(x)$ for all $x \in \mathbb{R}^n$ and $|\alpha| \leq d$, uniformly on compact subsets of $\mathbb{R}^n$. From (3.7) it follows that $p_k^{i(x_i)} \to p_k^{(x)}$ uniformly on compact sets whenever $p^i \to p$ in coefficients and $x_i \to x \in \mathbb{R}^n$. Thus, for every $0 \leq k \leq d$,

$$\|p_k^{i(x_i)}\|_{L^\infty(B_{r_i})} \to \|p_k^{(x)}\|_{L^\infty(B_r)} \tag{3.16}$$

whenever $p^i \to p$ in coefficients, $x_i \to x \in \mathbb{R}^n$ and $r_i \to r \in (0, \infty)$. We conclude that

$$\max_{j \neq k} \frac{\|p_j^{i(x_i)}\|_{L^\infty(B_{r_i})}}{\|p_k^{i(x_i)}\|_{L^\infty(B_{r_i})}} \to \max_{j \neq k} \frac{\|p_j^{(x)}\|_{L^\infty(B_r)}}{\|p_k^{(x)}\|_{L^\infty(B_r)}} \in [0, \infty]. \tag{3.17}$$

(Note $0/0$ never appears in (3.17) because the polynomials p^i and p are not identically zero.) That is, $\zeta_k(p^i, x_i, r_i) \to \zeta_k(p, x, r)$ whenever $p^i \to p$ in coefficients, $x_i \to x \in \mathbb{R}^n$ and $r_i \to r \in (0, \infty)$, as desired. □

Remark 3.12. If $(p^i)_{i=1}^{\infty}$ is a sequence of polynomials in $\mathbb{R}^n$ such that $d = \max_i p^i < \infty$, then $p^i \to p$ in coefficients if and only if $p^i \to p$ uniformly on compact sets.

Next we show that the relative size of the linear term of a polynomial decays linearly at any root of the polynomial.

Lemma 3.13. *If $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$ and $p(x) = 0$, then $\zeta_1(p, x, sr) \leq s\zeta_1(p, x, r)$ for all $r > 0$ and $s \in (0, 1)$.*

Proof. Suppose $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree $d \geq 1$. First if $d = 1$ and $p(x) = 0$, then $p = p_1^{(x)}(\cdot - x)$ and $\zeta_1(p, x, r) = 0$ for all $r > 0$ by Lemma 3.9. Second if $d \geq 2$ and $p(x) = 0$, then $p = p_d^{(x)}(\cdot - x) + \cdots + p_2^{(x)}(\cdot - x) + p_1^{(x)}(\cdot - x)$. Thus

$$\begin{aligned} \zeta_1(p, x, sr) = \max_{j>1} \frac{\|p_j^{(x)}\|_{L^\infty(B_{sr})}}{\|p_1^{(x)}\|_{L^\infty(B_{sr})}} &= \max_{j>1} \frac{s^j \|p_j^{(x)}\|_{L^\infty(B_r)}}{s\|p_1^{(x)}\|_{L^\infty(B_r)}} \\ &\leq \max_{j>1} \frac{s^2 \|p_j^{(x)}\|_{L^\infty(B_r)}}{s\|p_1^{(x)}\|_{L^\infty(B_r)}} = s\zeta_1(p, x, r) \end{aligned} \tag{3.18}$$

for all $r > 0$ and $s \in (0, 1)$. □

Now let us specialize to harmonic polynomials.

Lemma 3.14. *If h is a harmonic polynomial in $\mathbb{R}^n$ (i.e. $\Delta h = 0$) of degree $d \geq 1$, then $h_k^{(x)}$ is harmonic for all $0 \leq k \leq d$ and $x \in \mathbb{R}^n$.*

Proof. Suppose that h is a harmonic polynomial of degree $d \geq 1$ and let $x \in \mathbb{R}^n$. Applying Laplace's operator to (3.5) yields

$$0 = \Delta h_d^{(x)} + \Delta h_{d-1}^{(x)} + \cdots + \Delta h_2^{(x)}. \tag{3.19}$$

Since $\Delta h_k^{(x)}$ is the sum of monomials of degree $k - 2$ for each non-zero $h_k^{(x)}$, the right hand side of (3.19) vanishes if and only if $\Delta h_k^{(x)} = 0$ for all $0 \leq k \leq d$. □

Remark 3.15. If $h : \mathbb{R}^n \to \mathbb{R}$ is any harmonic polynomial of degree $d \geq 1$, then

$$\zeta_k(h, x, r) = \max_{j \neq k} \frac{\|h_j^{(x)}\|_{L^\infty(\partial B_r)}}{\|h_k^{(x)}\|_{L^\infty(\partial B_r)}} \tag{3.20}$$

by Lemma 3.14 and the maximum principle for harmonic functions.

3.2 Inequalities for Spherical Harmonics

Definition 3.16. A *spherical harmonic* $h : S^{n-1} \to \mathbb{R}$ of *degree* k is the restriction of a homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree k to the unit sphere.

Remark 3.17. If $h : S^{n-1} \to \mathbb{R}$ is a spherical harmonic, then there may exist distinct polynomials $p : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$ such that $p|_{S^{n-1}} = q|_{S^{n-1}} = h$. For instance, the polynomials $p(x) = 1$ and $q(x) = |x|^2 = x_1^2 + \cdots + x_n^2$ agree on S^{n-1}. Nevertheless, there always exists a unique (homogeneous) harmonic polynomial $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ such that $\tilde{h}|_{S^{n-1}} = h$.

Starting from well-known local estimates for the derivatives of harmonic functions on B_2 at points of S^{n-1}, we derive several inequalities for spherical harmonics.

Lemma 3.18. *If u is a real-valued harmonic function on $B_2 = \overline{B(0,2)}$, then*

$$|D^\alpha u(\theta)| \leq (2^{n+1} n|\alpha|)^{|\alpha|} \|u\|_{L^\infty(\partial B_2)} \quad \text{for every } \theta \in S^{n-1} \text{ and multi-index } \alpha. \tag{3.21}$$

Proof. For example, by Theorem 7 in §2.2 of [8] with $r = 1$,

$$|D^\alpha u(\theta)| \leq \frac{(2^{n+1} n|\alpha|)^{|\alpha|}}{\omega_n} \|u\|_{L^1(B(\theta,1))} \tag{3.22}$$

where $\omega_n = \mathcal{L}^n(B(0,1))$ denotes the volume of the unit ball in $\mathbb{R}^n$. Thus $\|u\|_{L^1(B(\theta,1))} \leq \omega_n \|u\|_{L^\infty(B(\theta,1))} \leq \omega_n \|u\|_{L^\infty(\partial B_2)}$, where the last inequality holds by the maximum principle for harmonic functions. □

Uniformly bounded spherical harmonics of degree k have a uniform Lipschitz constant.

Proposition 3.19 ([3] Proposition 3.2)**.** *For every $n \geq 2$ and $k \geq 1$ there exists a constant $A_{n,k} > 1$ such that for every spherical harmonic $h : S^{n-1} \to \mathbb{R}$ of degree k,*

$$|h(\theta_1) - h(\theta_2)| \leq A_{n,k} \|h\|_{L^\infty(S^{n-1})} |\theta_1 - \theta_2| \quad \text{for all } \theta_1, \theta_2 \in S^{n-1}. \tag{3.23}$$

Proof. Let $h : S^{n-1} \to \mathbb{R}$ be a spherical harmonic of degree k. By Remark 3.17, we may identify h with a homogeneous harmonic polynomial h of degree k defined on all of $\mathbb{R}^n$. Write $M = \|h\|_{L^\infty(S^{n-1})}$. If $|\theta_1 - \theta_2| \geq 1$ then $|h(\theta_1) - h(\theta_2)| \leq 2M \leq 2M|\theta_1 - \theta_2|$. Otherwise, suppose that $|\theta_1 - \theta_2| \leq 1$. By Lemma 3.18,

$$|D^\alpha h(\theta_2)| \leq (2^{n+1}n|\alpha|)^{|\alpha|}\|h\|_{L^\infty(\partial B_2)} = (2^{n+1}n|\alpha|)^{|\alpha|}2^k M \leq (2^{n+2}nk)^k M \tag{3.24}$$

for every multi-index α with $|\alpha| \leq k$, where $\|h\|_{L^\infty(\partial B_2)} = 2^k M$ since h is homogeneous of degree k. Expanding h in a Taylor series about θ_2,

$$h(\theta) - h(\theta_2) = \sum_{1\leq|\alpha|\leq k} \frac{D^\alpha h(\theta_2)}{\alpha!}(\theta - \theta_2)^\alpha. \tag{3.25}$$

Evaluating (3.25) at $\theta = \theta_1$ and applying the estimate (3.24),

$$|h(\theta_1) - h(\theta_2)| \leq \sum_{1\leq|\alpha|\leq k} \frac{(2^{n+2}nk)^k M}{\alpha!}\,|(\theta_1 - \theta_2)^\alpha| \leq A_{n,k}M|\theta_1 - \theta_2| \tag{3.26}$$

where $A_{n,k} = (2^{n+2}nk)^k \sum_{1\leq|\alpha|\leq k}(\alpha!)^{-1}$. (We used the fact $|x^\alpha| \leq |x|$ if $|x| \leq 1$.) □

We now derive several inequalities from Proposition 3.19.

Corollary 3.20. *For every spherical harmonic $h : S^{n-1} \to \mathbb{R}$ of degree $k \geq 1$,*

$$|h(\theta)| \leq A_{n,k}\|h\|_{L^\infty(S^{n-1})} \operatorname{dist}(\theta, \Sigma_h) \tag{3.27}$$

Proof. Apply Proposition 3.19 with $\theta_1 = \theta$ and $\theta_2 \in \Sigma_h \cap S^{n-1}$. Then minimizing (3.23) over $\theta_2 \in \Sigma_h \cap S^{n-1}$ yields (3.27). □

The next inequality roughly says that a spherical harmonic takes its "big values" on a "big piece" of the unit sphere.

Corollary 3.21 ([3] Corollary 3.3)**.** *For every $n \geq 2$ and $k \geq 1$ there exists a constant $l_{n,k} > 0$ such that for every spherical harmonic $h : S^{n-1} \to \mathbb{R}$ of degree k,*

$$\mathcal{H}^{n-1}\{\theta \in S^{n-1} : |h(\theta)| \geq \tfrac{1}{2}\|h\|_{L^\infty(S^{n-1})}\} \geq l_{n,k}. \tag{3.28}$$

Proof. Choose $\theta_0 \in S^{n-1}$ such that $|h(\theta_0)| = \|h\|_{L^\infty(S^{n-1})} = M$. By Proposition 3.19,

$$|h(\theta)| \geq |h(\theta_0)| - |h(\theta) - h(\theta_0)| \geq M(1 - A_{n,k}|\theta - \theta_0|). \tag{3.29}$$

If $|\theta - \theta_0| \leq 1/2A_{n,k}$, then $|h(\theta)| \geq M/2$. That is, the set $\{\theta \in S^{n-1} : |h(\theta)| \geq M/2\}$ contains the surface ball $\Delta(\theta_0, 1/2A_{n,k})$. Thus $l_{n,k} = \mathcal{H}^{n-1}(\Delta(\theta_0, 1/2A_{n,k}))$ suffices. □

Thus the spherical harmonics of degree k satisfy a reverse Hölder inequality.

Corollary 3.22 ([3] Corollary 3.4)**.** *For every $n \geq 2$ and $k \geq 1$ there exists a constant $B_{n,k} > 1$ such that for every spherical harmonic $h : S^{n-1} \to \mathbb{R}$ of degree k,*

$$\|h\|_{L^\infty(S^{n-1})} \leq B_{n,k}\|h\|_{L^1(S^{n-1})}. \tag{3.30}$$

Proof. Let $\Gamma = \{\theta \in S^{n-1} : |h(\theta)| \geq \frac{1}{2}\|h\|_{L^\infty(S^{n-1})}\}$. By Corollary 3.21,

$$\|h\|_{L^1(S^{n-1})} \geq \frac{1}{2}\|h\|_{L^\infty(S^{n-1})}\sigma(\Gamma) \geq \frac{l_{n,k}}{2}\|h\|_{L^\infty(S^{n-1})} \tag{3.31}$$

and $B_{n,k} = 2/l_{n,k}$ suffices. □

3.3 Convergence of Zero Sets

Next we discuss the relationship between convergence of polynomials in coefficients and convergence of their zero sets in the Hausdorff distance. To start, we give a cautionary example which shows how and where issues can arise. Recall $B(x, r)$ always denotes the *closed* ball with center $x \in \mathbb{R}^n$ and radius $r > 0$.

Example 3.23. Let $h(x, y) = xy$ and for each $i \geq 1$ let $h^i(x, y) = h(x+1/i, y) = xy+y/i$. Then the polynomials h and h^i ($i \geq 1$) are harmonic and $h^i \to h$ in coefficients. However, we claim there exists a closed ball B such that $\Sigma_{h^i} \cap B \neq \emptyset$ for all $i \geq 1$ and $\Sigma_h \cap B \neq \emptyset$ but $\Sigma_{h^i} \cap B$ does not converge to $\Sigma_h \cap B$ in the Hausdorff distance (see Figure 3.6). Indeed let $B = B((1, 1/2), 1)$. Then $\Sigma_{h^i} \cap B = [1 - c, 1 + c] \times \{0\}$, $c = \sqrt{3}/2$ is a fixed line segment for all $i \geq 1$, but $\Sigma_h \cap B = ([1 - c, 1 + c] \times \{0\}) \cup \{(0, 1/2)\}$ consists of the line segment together with an additional point of ∂B. Thus convergence in coefficients does not

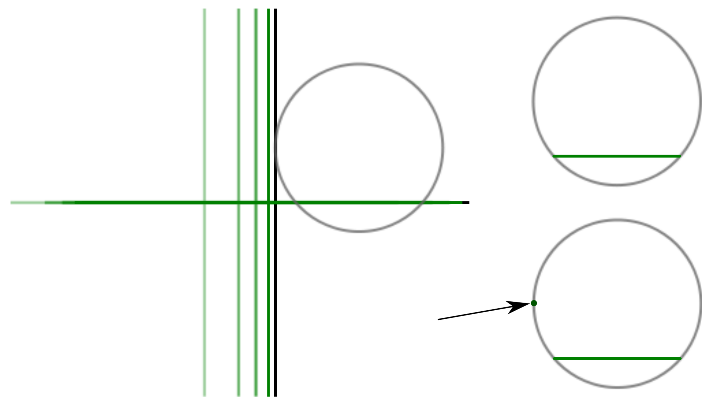

Figure 3.6: Convergence in Coefficients versus Convergence of Zero Sets

imply (local) convergence of the zero sets in the Hausdorff distance in general. Note that the spurious point lies on ∂B and is isolated in $\Sigma_h \cap B$, even though it is not isolated in Σ_h.

Lemma 3.24. *If $(h_i)_{i=1}^{\infty}$ is a sequence of harmonic polynomials and $h^i \to h$ in coefficients, then h is a harmonic polynomial. Moreover, if h is nonconstant and if $\Sigma_h \cap \operatorname{int} B \neq \emptyset$ for some closed ball B, then $\Sigma_{h^i} \cap B \neq \emptyset$ for all sufficiently large i. Furthermore, if $\Sigma_{h^i} \cap B$ converges in the Hausdorff distance to a closed set $F \subset B$, then $F \subset \Sigma_h \cap B$ and $F \cap \operatorname{int} B = \Sigma_h \cap \operatorname{int} B$.*

Proof. Suppose that $h^i : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial for each $i \geq 1$ and $h^i \to h$ in coefficients. Then $h^i \to h$ uniformly on compact subsets of $\mathbb{R}^n$. Hence h is harmonic.

Now suppose that $d = \max_{i \geq 1} \deg h^i = \deg h \geq 1$ and $\Sigma_h \cap \operatorname{int} B \neq \emptyset$ for some closed ball B. Since $\Sigma_h \cap \operatorname{int} B \neq \emptyset$, we can find a ball $B' \subset B$ whose center lies in $\Sigma_h \cap B$. By the mean value property,

$$\int_{B'} h(y)dy = 0. \tag{3.32}$$

Because h is not identically zero, there must exist $x_+, x_- \in B'$ such that $h(x_+) > 0$ and $h(x_-) < 0$. Hence, since $h^i(x_\pm) \to h(x_\pm)$, we conclude $h^i(x_+) > 0$ and $h^i(x_-) < 0$ for all sufficiently large i, as well. By the intermediate value theorem, h^i must vanish somewhere in $B' \subset B$ for all large i. That is, $\Sigma_{h^i} \cap B \neq \emptyset$ for all sufficiently large i.

Finally suppose that $\Sigma_{h^i} \cap B \to F$ in the Hausdorff distance for some closed set F. Recall we want to show that $F \subset \Sigma_h \cap B$ and $F \cap \operatorname{int} B = \Sigma_h \cap \operatorname{int} B$. On one hand, for every $y \in F$ there exists $y_i \in B$ such that $h^i(y_i) = 0$ and $y_i \to y$. To show $h(y) = 0$, consider

$$\begin{aligned} h(y) &= h(y) - h(y_i) + h(y_i) - h^i(y_i) + h^i(y_i) \\ &= h(y) - h(y_i) + h(y_i) - h^i(y_i). \end{aligned} \tag{3.33}$$

Since h is continuous and $y_i \to y$, we get $\limsup_{i\to\infty} |h(y) - h(y_i)| = 0$. Because $h^i \to h$ uniformly on B, we conclude $\limsup_{i\to\infty} |h(y_i) - h^i(y_i)| \leq \limsup_{i\to\infty} \|h - h^i\|_{L^\infty(B)} = 0$. Hence $h(y) = 0$ and $F \subset \Sigma_h \cap B$. In particular, $F \cap \operatorname{int} B \subset \Sigma_h \cap \operatorname{int} B$.

On the other hand, suppose that $y \in \Sigma_h \cap \operatorname{int} B$. Choose $m \geq 1$ such that $B(y, 1/m) \subset \operatorname{int} B$. Since h is not identically zero, we can use the mean value property of h as above to show there exist $y_+^m, y_-^m \in B(y, 1/m)$ such that $h(y_+^m) > 0$ and $h(y_-^m) < 0$. But $h^i(y_\pm^m) \to h(y_\pm^m)$, so there exists i_0 such that $h^i(y_+^m) > 0$ and $h^i(y_-^m) < 0$ for all $i \geq i_0$. By continuity, we conclude that for each $i \geq i_0$ there exists $y_0^{m,i} \in B(y, 1/m)$ such that $h^i(y_0^{m,i}) = 0$. Thus

$$\begin{aligned} \operatorname{dist}(y, F) &\leq \limsup_{i\to\infty} \operatorname{dist}(y, \Sigma_{h^i} \cap B) + \operatorname{HD}(\Sigma_{h^i} \cap B, F) \\ &\leq \frac{1}{m} + \limsup_{k\to\infty} \operatorname{HD}(\Sigma_{h^i} \cap B, F) = \frac{1}{m}. \end{aligned} \tag{3.34}$$

Letting $m \to \infty$ yields $\operatorname{dist}(y, F) = 0$. Since F is closed, this implies that $y \in F$ and $\Sigma_h \cap \operatorname{int} B \subset F \cap \operatorname{int} B$. Therefore, $F \cap \operatorname{int} B = \Sigma_h \cap B$, as desired. □

Corollary 3.25. *Suppose that $(h^i)_{i=1}^\infty$ is a sequence of harmonic polynomials and $h^i \to h$ in coefficients. If h is nonconstant and B is a closed ball such that $\Sigma_h \cap \operatorname{int} B \neq \emptyset$ and*

$$\Sigma_h \cap B = \overline{\Sigma_h \cap \operatorname{int} B}, \tag{3.35}$$

then $\Sigma_{h^i} \cap B \to \Sigma_h \cap B$ in the Hausdorff distance.

Proof. Let $(h^i)_{i=1}^\infty$ be any sequence of harmonic polynomials in $\mathbb{R}^n$ such that $h^i \to h$ in coefficients to a nonconstant harmonic polynomial h. Suppose B is a closed ball such that

$\Sigma_h \cap \operatorname{int} B \neq \emptyset$ and such that (3.35) holds. By Lemma 3.24, $\Sigma_{h^i} \cap B \neq \emptyset$ for all sufficiently large i. Pick an arbitrary subsequence $(h^{ij})_{j=1}^{\infty}$ of $(h^i)_{i=1}^{\infty}$. By Blaschke's selection theorem (Lemma 2.18), we can find a further subsequence $(h^{ijk})_{k=1}^{\infty}$ of $(h^{ij})_{j=1}^{\infty}$ and a nonempty closed set $F \subset B$ such that $\lim_{k\to\infty} \mathrm{HD}(\Sigma_{h^{ijk}} \cap B, F) = 0$. By Lemma 3.24, $F \subset \Sigma_h \cap B$ and $F \cap \operatorname{int} B = \Sigma_h \cap \operatorname{int} B$. Thus, since B satisfies (3.35) and F is closed,

$$\Sigma_h \cap B = \overline{\Sigma_h \cap \operatorname{int} B} = \overline{F \cap \operatorname{int} B} \subset \overline{F} = F \tag{3.36}$$

This shows $F = \Sigma_h \cap B$. We have proved every subsequence $(h^{ij})_{j=1}^{\infty}$ of $(h^i)_{i=1}^{\infty}$ has a further subsequence $(h^{ijk})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \mathrm{HD}(\Sigma_{h^{ijk}} \cap B, \Sigma_h \cap B) = 0$. Therefore, the original sequence $\Sigma_{h^i} \cap B$ also converges to $\Sigma_h \cap B$ in the Hausdorff distance. □

Corollary 3.26. *Suppose that $h : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial of degree $d \geq 1$, and let $x \in \Sigma_h$. If $h(y) = h_d^{(x)}(y-x) + h_{d-1}^{(x)}(y-x) + \cdots + h_j^{(x)}(y-x)$ where $h_j^{(x)} \neq 0$, then the unique blow-up of Σ_h at x is the zero set $\Sigma_{h_j^{(x)}}$ of $h_j^{(x)}$. That is,*

$$\lim_{r_i \to 0} \mathrm{HD}\left(\frac{\Sigma_h - x}{r_i} \cap B_s, \Sigma_{h_j^{(x)}} \cap B_s\right) = 0 \quad \textit{for all } s > 0. \tag{3.37}$$

Proof. Suppose that $h(y) = h_d^{(x)}(y-x) + \cdots + h_j^{(x)}(y-x)$ is a harmonic polynomial in $\mathbb{R}^n$ with $j \geq 1$ and $h_j^{(x)} \neq 0$. Given $r_i \downarrow 0$, define $h^i(y) = r_i^{-j} h(x + r_i y)$ for all $y \in \mathbb{R}^n$. Then $h^i : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial and $r_i^{-1}(\Sigma_h - x) = \Sigma_{h^i}$. Moreover,

$$h^i(y) = r_i^{-j}\left(h_d^{(x)}(r_i y) + \cdots + h_j^{(x)}(r_i y)\right) = r_i^{d-j} h_d^{(x)}(y) + \cdots + h_j^{(x)}(y). \tag{3.38}$$

Since $r_i \to 0$ as $i \to \infty$, $h^i \to h_j^{(x)}$ in coefficients. Because $h_j^{(x)}$ is homogeneous, $\Sigma_{h_j^{(x)}} \cap B_s$ satisfies (3.35). Thus the claim follows immediately from Corollary 3.25. □

Remark 3.27. The proofs of Lemma 3.24 and Corollary 3.25 did not use the full strength of the harmonic property of h. Instead we only needed to assume that h_i are polynomials, $h_i \to h$ in coefficients, and h is a nonconstant polynomial such that for all $x \in \Sigma_h$ and for all $r > 0$ there exist $x_+, x_- \in B(x, r)$ such that $h(x_+) > 0$ and $h(x_-) < 0$.

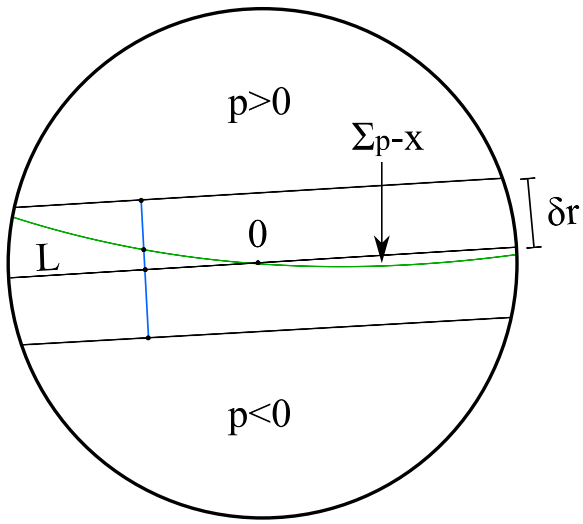

Figure 3.7: Proof of Lemma 3.28

3.4 Local Flatness of Zero Sets

How flat is the zero set Σ_h of a harmonic polynomial h? In order to answer this question, we show that the flatness $\theta_{\Sigma_h}(x,r)$ of Σ_h in $B(x,r)$ is comparable to the relative size $\zeta_1(h,x,r)$ of the linear term $h_1^{(x)}$. The first lemma is valid for all polynomials p.

Lemma 3.28. *If $p:\mathbb{R}^n\rightarrow\mathbb{R}$ is a polynomial of degree $d\geq 2$ such that $p(x)=0$, then $\theta_{\Sigma_p}(x,r)\leq\sqrt{5}(d-1)\zeta_1(p,x,r)$ for all $r>0$.*

Proof. Let $p:\mathbb{R}^n\rightarrow\mathbb{R}$ be any polynomial of degree $d\geq 2$, let $x\in\Sigma_p$ and let $r>0$. For the proof we may assume that

$$\sqrt{5}(d-1)\zeta_1(p,x,r)\leq 2, \tag{3.39}$$

since the bound $\theta_{\Sigma_p}(x,r)\leq 2$ is always true. Because $\zeta_1(p,x,r)<\infty$, we know that $p_1^{(x)}\neq 0$, by Lemma 3.9. Hence $L=\{p_1^{(x)}=0\}\in G(n,n-1)$ is an $(n-1)$-dimensional plane through the origin. Let e be the unique unit normal vector to L at 0 such that $p_1^{(x)}(e)>0$, and set $\delta := (d-1)\zeta_1(p,x,r)$. If $y \in L$ and $t > \delta$ satisfies $|y + tre| \leq r$, then

$$\begin{aligned} p(x+y+tre) &= p_d^{(x)}(y+tre) + \cdots + p_1^{(x)}(y+tre) \\ &\geq p_1^{(x)}(y+tre) - \|p_d^{(x)}\|_{L^\infty(B_r)} - \cdots - \|p_2^{(x)}\|_{L^\infty(B_r)} \\ &\geq p_1^{(x)}(y+tre) - (d-1)\zeta_1(p,x,r)\|p_1^{(x)}\|_{L^\infty(B_r)} \\ &= t\|p_1^{(x)}\|_{L^\infty(B_r)} - \delta\|p_1^{(x)}\|_{L^\infty(B_r)} > 0. \end{aligned} \tag{3.40}$$

Similarly, $p(x+y+tre) < 0$ when $|y+tre| \leq r$ and $t < -\delta$. Hence every root of p in $B(x,r)$ lies in the strip $\{x+y+tre : y \in L \text{ and } |t| \leq \delta\}$. Thus $\operatorname{dist}(z, x+L) \leq \delta r$ for all $z \in \Sigma_p \cap B(x,r)$.

On the other hand, suppose that $y \in L \cap B_r$. Then we can connect y by two line segments $\ell_\pm = [y, z_\pm]$ to $(L \pm \delta re) \cap B_r$ of minimal length (see Figure 3.7). Since $p(x+z_+) \geq 0$ and $p(x+z_-) \leq 0$, by continuity $p(x+z_0)$ must vanish at some point $z_0 \in \ell_+ \cup \ell_-$. Hence $\operatorname{dist}(x+y, \Sigma_p \cap B(x,r))$ is bounded above by the length of $\ell_\pm$. This is a geometric constant, which is at worst $2r(1-\sqrt{1-\delta^2})$. (To compute this, notice the length of $\ell_\pm$ is at worst the distance of $(r, \overline{0}, 0)$ to $(tr, \overline{0}, \delta r)$ where $t^2 + \delta^2 = 1$.) The assumption (3.39) yields $2r(1-\sqrt{1-\delta^2}) \leq (\sqrt{5}-1)\delta r$. Thus, $\operatorname{dist}(x+y, \Sigma_p \cap B(x,r)) \leq (\sqrt{5}-1)\delta r$ for all $x+y \in x+L$. Therefore, $\theta_{\Sigma_p}(x,r) \leq \sqrt{5}\delta = \sqrt{5}(d-1)\zeta_1(p,x,r)$, as desired. □

The converse of Lemma 3.28 does not hold in general. Indeed if $p(x,y) = x^4 + y^4 - y^2$, then $\zeta_1(p,0,r) = \infty$ for all $r > 0$ even though $\lim_{r\to 0} \theta_{\Sigma_p}(0,r) = 0$. Nevertheless we can establish a converse to Lemma 3.28 for harmonic polynomials! As an intermediate step, we first show that the zero sets of homogeneous harmonic polynomials of degree $k \geq 2$ are uniformly not flat at the origin.

Lemma 3.29. *For all $n \geq 2$ and $k \geq 2$ there exists a constant $\delta'_{n,k} > 0$ such that*

$$\theta_{\Sigma_h}(0,r) \geq \delta'_{n,k} \quad \textit{for all } r > 0 \tag{3.41}$$

for every homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree k.

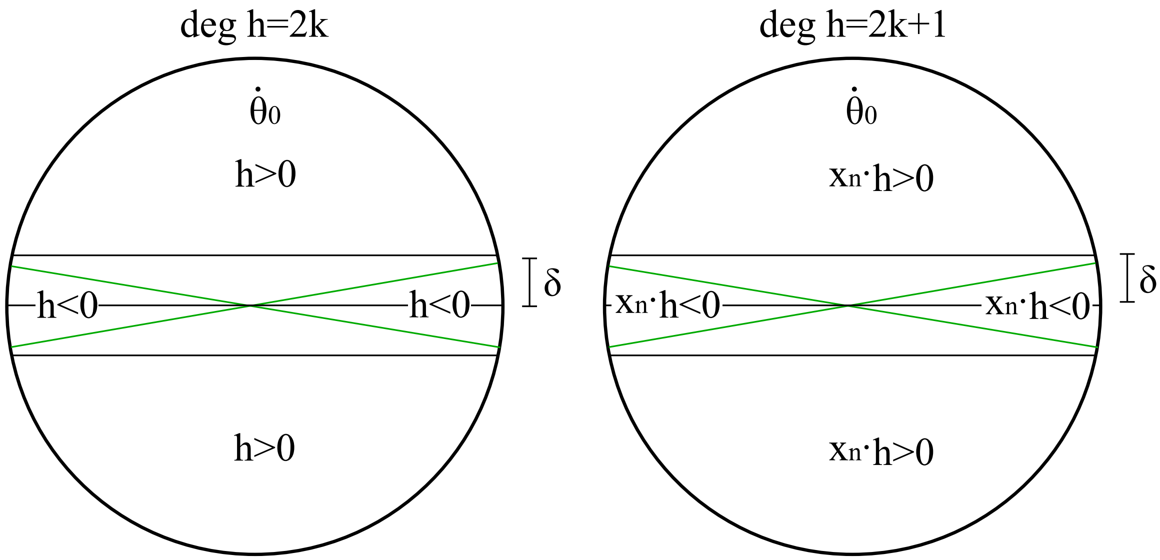

Figure 3.8: Proof of Lemma 3.29

Proof. Suppose that $h : \mathbb{R}^n \to \mathbb{R}$ is a homogeneous harmonic polynomial of degree k. Since h is homogeneous, $\delta = \theta_{\Sigma_h}(0,1) = \theta_{\Sigma_h}(0,r)$ for all $r > 0$. By applying a rotation, we may assume without loss of generality that $\mathrm{HD}(\Sigma_h \cap B(0,1), \{x_n = 0\} \cap B(0,1)) \leq \delta$. Also, by replacing h with $-h$ if necessary, we may assume that there exists $\theta_0 \in S^{n-1}$ such that $h(\theta_0) = \|h\|_{L^\infty(S^{n-1})}$ (i.e. the supremum norm is obtained at a positive value of h). Finally, by performing a change of coordinates $x \mapsto -x$ if necessary, we may assume $\theta_0 \in \mathbb{R}^{n-1} \times \mathbb{R}^+$ (i.e. the last coordinate of θ_0 is positive). We now break the argument into two cases, depending on the parity of k.

Suppose that $k \geq 2$ is even. By the mean value property for harmonic functions,

$$\frac{1}{\sigma_{n-1}} \int_{S^{n-1}} h(\theta) d\mathcal{H}^{n-1}(\theta) = h(0) = 0. \tag{3.42}$$

We will show that δ being small violates (3.42). By Corollary 3.20, $\mathrm{dist}(\theta_0, \Sigma_h) \geq A_{n,k}^{-1}$. Hence $h(\theta) > 0$ for all $\theta \in S^{n-1} \cap \{x_n > \delta\}$ provided $\delta \ll A_{n,k}^{-1}$ (the last coordinate of θ_0 is positive). Assume this is true. Since h is even, $h(\theta) > 0$ for all $\theta \in S^{n-1} \cap \{x_n < -\delta\}$, as well. Thus negative values of h (obtained at points of the sphere) can only be obtained inside the strip $S_\delta = S^{n-1} \cap \{|x_n| \le \delta\}$. Moreover, $|h(\theta)| \le 2\delta A_{n,k} \|h\|_{L^\infty(S^{n-1})}$ for $\theta \in S_\delta$, by Corollary 3.20. But $h(\theta) \ge (1/2)\|h\|_{L^\infty(S^{n-1})}$ for all $\theta \in \Delta_0 = S^{n-1} \cap B(\theta_0, 1/2A_{n,k})$ by Proposition 3.19 (also see the proof of Corollary 3.21). It follows that

$$\begin{aligned}
\int_{S^{n-1}} h(\theta) d\mathcal{H}^{n-1}(\theta) &= \int_{S^{n-1}} h^+(\theta) d\mathcal{H}^{n-1}(\theta) - \int_{S^{n-1}} h^-(\theta) d\mathcal{H}^{n-1}(\theta) \\
&\ge \int_{\Delta_0} h(\theta) d\mathcal{H}^{n-1}(\theta) - \int_{S_\delta} |h(\theta)| d\mathcal{H}^{n-1}(\theta) \\
&\ge \|h\|_{L^\infty(S^{n-1})} \left(\frac{1}{2}\mathcal{H}^{n-1}(\Delta_0) - 2\delta A_{n,k} \mathcal{H}^{n-1}(S_\delta) \right) > 0
\end{aligned} \tag{3.43}$$

if δ is too small, for example if $\delta < \mathcal{H}^{n-1}(\Delta_0)/8A_{n,k}\sigma_{n-1} = \delta'_{n,k}$, which violates (3.42). Therefore, $\delta \ge \delta'_{n,k}$ when k is even.

Suppose that $k \ge 3$ is odd. Because the spherical harmonics of different degrees are orthogonal in $L^2(S^{n-1})$ (e.g. see [2] Proposition 5.9),

$$\int_{S^{n-1}} \theta_n h(\theta) d\mathcal{H}^{n-1}(\theta) = 0. \tag{3.44}$$

This time we will show that (3.44) is violated if δ is small. Since $\operatorname{dist}(\theta_0, \Sigma_h) \ge A_{n,k}^{-1}$, $h(\theta) > 0$ and $\theta_n h(\theta) > 0$ for all $\theta \in S^{n-1} \cap \{x_n > \delta\}$ if $\delta \ll A_{n,k}^{-1}$. Assume this is true. Since h is odd, $\theta_n h(\theta)$ is even and $\theta_n h(\theta) > 0$ for all $\theta \in S^{n-1} \cap \{x_n < -\delta\}$ too. Hence $\theta_n h(\theta)$ can only assume negative values in the strip $S_\delta = S^{n-1} \cap \{|x_n| \le \delta\}$. Moreover, $|\theta_n h(\theta)| \le 2\delta^2 A_{n,k} \|h\|_{L^\infty(S^{n-1})}$ for every $\theta \in S_\delta$, by Corollary 3.20. On the other hand, $\theta_n h(\theta) > \delta(1/2)\|h\|_{L^\infty(S^{n-1})}$ for all $\theta \in \Delta_0 = S^{n-1} \cap B(\theta_0, 1/2A_{n,k})$, by Proposition 3.19. Thus

$$\begin{aligned}
\int_{S^{n-1}} \theta_n h(\theta) d\mathcal{H}^{n-1}(\theta) &= \int_{S^{n-1}} (\theta_n h(\theta))^+ d\mathcal{H}^{n-1}(\theta) - \int_{S^{n-1}} (\theta_n h(\theta))^- d\mathcal{H}^{n-1}(\theta) \\
&\ge \int_{\Delta_0} \theta_n h(\theta) d\mathcal{H}^{n-1}(\theta) - \int_{S_\delta} |\theta_n h(\theta)| d\mathcal{H}^{n-1}(\theta) \\
&\ge \delta \|h\|_{L^\infty(S^{n-1})} \left(\frac{1}{2}\mathcal{H}^{n-1}(\Delta_0) - 2\delta A_{n,k} \mathcal{H}^{n-1}(S_\delta) \right) > 0
\end{aligned} \tag{3.45}$$

if δ is too small, for example if $\delta < \mathcal{H}^{n-1}(\Delta_0)/8A_{n,k}\sigma_{n-1} = \delta'_{n,k}$, which violates (3.44). Therefore, $\delta \ge \delta'_{n,k}$ when k is odd. $\square$

Harmonic polynomials enjoy a partial converse to Lemma 3.28.

Proposition 3.30. *For all $n \geq 2$ and $d \geq 2$ there exist $\delta_{n,d} > 0$ with the following property. If $h : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial of degree d and $h(x) = 0$, then $\zeta_1(h, x, r) < \delta_{n,d}^{-1}$ whenever $\theta_{\Sigma_h}(x, r) < \delta_{n,d}$.*

Proof. Let $n \geq 2$ and $d \geq 2$ be fixed. Suppose for contradiction that for every $N \geq 1$ there exists a harmonic polynomial $h^N : \mathbb{R}^n \to \mathbb{R}$ of degree d, $x_N \in \mathbb{R}^n$ and $r_N > 0$ such that $h^N(x_N) = 0$, $\zeta_1(h^N, x_N, r_N) > N$ and $\theta_{\Sigma_{h^N}}(x_N, r_N) < N$. Replacing each polynomial h^N with $\tilde{h}^N(y) = c_N h(r_N(y + x_N))$, we may assume without loss of generality that $x_N = 0$ and $r_N = 1$ for all $N \geq 1$, and $\max_{|\alpha| \leq d} |D^\alpha h^N(0)| = 1$. Thus, there exists a sequence h^N of harmonic polynomials in $\mathbb{R}^n$ of degree d with uniformly bounded coefficients such that $h^N(0) = 0$, $\zeta_1(h^N, 0, 1) \geq N$ and $\theta_{\Sigma_{h^N}}(0, 1) \leq 1/N$ for all $N \geq 1$. Passing to a subsequence, we may assume that $h^N \to h$ in coefficients to some harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$. Also note h is nonconstant because we assumed for each h^N there is some multi-index α such that $|D^\alpha h^N(0)| = 1$. Hence $\zeta_1(h, 0, 1) = \infty$, by Lemma 3.11. Taking a further subsequence we may also assume that there exists a closed set F such that $\Sigma_{h^N} \cap B_1 \to F$ in the Hausdorff distance. By Lemma 3.24, $F \cap \operatorname{int} B_1 = \Sigma_h \cap \operatorname{int} B_1$. Hence $\theta_{\Sigma_h}(0, r) = 0$ for all $r < 1$. To complete the proof we blow up Σ_h at the origin and apply Lemma 3.29. Expand h as $h = h_k^{(0)} + \cdots + h_j^{(0)}$ where $k = \deg h$ and $h_j^{(0)} \neq 0$. Note $2 \leq j \leq k$, since $\zeta_1(h, 0, 1) = \infty$. Choose any sequence $r_i \downarrow 0$ and define $h^i(y) = h(r_i y)$ for all $y \in \mathbb{R}^n$. By Corollary 3.26, $\Sigma_{h^i} \cap B_1 \to \Sigma_{h_j^{(0)}} \cap B_1$ in the Hausdorff distance. Therefore, since $\theta_{\Sigma_{h^i}}(0, 1) = \theta_{\Sigma_h}(0, r_i) = 0$ for all i such that $r_i < 1$, we conclude that $\theta_{\Sigma_{h_j^{(0)}}}(0, 1) = 0$. Since $h_j^{(0)}$ is a homogeneous polynomial of degree $j \geq 2$, this contradicts Lemma 3.29. Our supposition was false. Hence there exists $N_{n,d} \geq 1$ such that for every harmonic polynomial h in $\mathbb{R}^n$ of degree d, every $x \in \Sigma_h$ and every $r > 0$, $\zeta_1(h, x, r) < N_{n,d}$ whenever $\theta_{\Sigma_h}(x, r) < 1/N_{n,d} = \delta_{n,d}$. □

Corollary 3.31. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial of degree $d \geq 2$. If $h(x) = 0$ and $\theta_{\Sigma_h}(x, r) < \delta_{n,d}$, then $Dh(x) \neq 0$ and $\theta_{\Sigma_h}(x, sr) < s\sqrt{5}(d-1)\delta_{n,d}^{-1}$ for all $s \in (0, 1)$.*

Proof. Assume that $h : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial of degree $d \geq 2$ such that $\theta_{\Sigma_h}(x, r) < \delta_{n,d}$ for some $x \in \Sigma_h$ and $r > 0$. Proposition 3.30 yields $\zeta_1(h, x, r) < \delta_{n,d}^{-1}$. Since $\zeta_1(h, x, r) < \infty$, Lemma 3.9 guarantees that $h_1^{(x)} \neq 0$, or equivalently, $Dh(x) \neq 0$. Moreover, by Lemma 3.13, $\zeta_1(h, x, sr) \leq s\zeta_1(h, x, r) < s\delta_{n,d}^{-1}$ for all $s \in (0, 1)$. Thus, by Lemma 3.28, $\theta_{\Sigma_h}(x, sr) < \sqrt{5}(d-1)s\delta_{n,d}^{-1}$ for all $s \in (0, 1)$. □

Corollary 3.32. *Let* $h : \mathbb{R}^n \to \mathbb{R}$ *be a harmonic polynomial of degree* $d \geq 2$*. If* $x \in \Sigma_h$ *and* $Dh(x) = 0$*, then* $\theta_{\Sigma_h}(x, r) \geq \delta_{n,d}$ *for all* $r > 0$*.*

Proof. This is the contrapositive of Corollary 3.31. □

Proof of Theorem 3.2. Suppose that $h : \mathbb{R}^n \to \mathbb{R}$, $n \geq 2$, is a harmonic polynomial of degree $d \geq 2$ and fix $x \in \Sigma_h$. If $Dh(x) \neq 0$, then $\zeta_1(h, x, 1) < \infty$ by Lemma 3.9. Applying Lemma 3.13 and Lemma 3.28, it follows that

$$\theta_{\Sigma_h}(x, r) \leq \sqrt{5}(d-1)r\zeta_1(h, x, 1) \quad \text{for all } r \in (0, 1). \tag{3.46}$$

Since $\zeta_1(h, x, 1) < \infty$, we see that $\theta_{\Sigma_h}(x, r) < \delta_{n,d}$ for some $r \in (0, 1)$ sufficiently small. Conversely, if $Dh(x) = 0$, then $\theta_{\Sigma_h}(x, r) \geq \delta_{n,d}$ for all $r > 0$, by Corollary 3.32. □

Remark 3.33. It is natural to ask if a stronger statement than Proposition 3.30 holds. Namely, is it true that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\theta_{\Sigma_h}(x, r) < \delta \Rightarrow \zeta_1(h, x, r) < \varepsilon$? Unfortunately the answer is no, as the following example illustrates. Consider the harmonic polynomial $h(x, y) = xy$ with root $(2, 0) \in \Sigma_h$ and $r = 1$. Since

$$\Sigma_h \cap B((2, 0), 1) = [1, 3] \times \{0\}$$

is a line segment, $\theta_{\Sigma_h}((2, 0), 1) = 0$. On the other hand, $h((x, y) + (2, 0)) = xy + 2y$. Hence $h_1^{(2,0)} = 2y$, $h_2^{(2,0)} = xy$ and $\zeta_1(h, (2, 0), 1) = \|xy\|_{L^\infty(B_1)}/\|2y\|_{L^\infty(B_1)} = 1/2 > 0$. This shows $\zeta_1(h, (2, 0), 1) > 0$ even though $\theta_{\Sigma_h}((2, 0), 1) < \delta$ for all $\delta > 0$.

Chapter 4

POLYNOMIAL HARMONIC MEASURES

For any harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree $d \geq 1$, the positive and negative parts $h^\pm$ of h are Green functions with pole at infinity for the unbounded open sets $\Omega_h^\pm = \{x \in \mathbb{R}^n : h^\pm(x) > 0\}$, whose common boundary is the zero set Σ_h of h.

Definition 4.1. The *harmonic measure ω_h associated to h* is the unique harmonic measure with pole *at infinity* on $\Omega_h^\pm$ with Green function $h^\pm$. That is,

$$\int_{\Sigma_h} \varphi \, d\omega_h = \int_{\Omega_h^+} h^+ \Delta\varphi = \int_{\Omega_h^-} h^- \Delta\varphi \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n). \tag{4.1}$$

Alternatively, by Theorem 3.1, the zero set Σ_h is C^1 away from a $(n-2)$-rectifiable subset. Hence there exists a unique outward unit normal $\nu^\pm$ on $\partial\Omega_h^\pm$ at almost every point with respect to the surface measure $\sigma = \mathcal{H}^{n-1} \llcorner \Sigma_h$ and (4.1) is equivalent to

$$d\omega_h = -\frac{\partial h^+}{\partial \nu^+} d\sigma = -\frac{\partial h^-}{\partial \nu^-} d\sigma \tag{4.2}$$

by the generalized Gauss-Green theorem (e.g., see Chapter 5 of Evans and Gariepy [9]). Observe that ω_h is a Radon measure, but unlike the harmonic measures of $\Omega_h^\pm$ with pole at $X \in \Omega_h^\pm$ is not a probability measure.

In this chapter, we focus on two collections of polynomial harmonic measures that arise as tangent measures of harmonic measure on 2-sided NTA domains in Chapter 5:

$$\mathcal{P}_d = \{\omega_h : h \text{ is a non-zero harmonic polynomial of degree} \leq d \text{ and } h(0) = 0\}, \tag{4.3}$$

$$\mathcal{F}_k = \{\omega_h : h \text{ is a homogenous harmonic polynomial of degree } k\}. \tag{4.4}$$

By convention we will use d for the degree of any non-zero polynomial, but reserve k for the degree of a homogeneous polynomial. If $1 \leq k \leq d$, note that $\mathcal{F}_k \subset \mathcal{P}_d$. When $k = 1$ the family $\mathcal{F}_1$ is the collection of $(n-1)$-flat measures in $\mathbb{R}^n$, i.e. Hausdorff measures restricted to codimension 1 hyperplanes through the origin.

Our first observation is that $\mathcal{P}_d$ and $\mathcal{F}_k$ fit into the framework of Chapter 2, §5.

Lemma 4.2 ([3] Lemma 4.1)**.** *$\mathcal{P}_d$ and $\mathcal{F}_k$ are dilation invariant cones.*

Proof. Suppose that ω_h is associated to a harmonic polynomial $h = h_d^{(0)} + \cdots + h_1^{(0)}$ and let $c, r > 0$. We claim $cT_{0,r}[\omega_h]$ is harmonic measure ω_g associated to $g(x) = cr^n h(rx)$, where $\Delta g = cr^{n+d}\Delta h_d^{(0)} + \cdots + cr^{n+2}\Delta h_2^{(0)} = 0$ by Lemma 3.14. For any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\Omega_g^+} g(x)\Delta\varphi(x)dx &= \int_{r^{-1}\Omega_h^+} cr^n h(rx)\Delta\varphi(x)dx = c\int_{\Omega_h^+} h(y)\Delta\varphi(r^{-1}y)dy \\ = c\int_{\Sigma_h} \varphi(r^{-1}y)d\omega_h(y) &= c\int_{r^{-1}\Sigma_h} \varphi(x)dT_{0,r}[\omega_h](x) = c\int_{\Sigma_g} \varphi(x)dT_{0,r}[\omega_h](x). \end{aligned} \tag{4.5}$$

Therefore, $\omega_g = cT_{0,r}[\omega_h]$, as claimed. Because the polynomial g has the same degree as h and g is homogeneous if h is homogeneous, $\mathcal{P}_d$ and $\mathcal{F}_k$ are dilation invariant cones. $\square$

Here is a practical formula to compute ω_h on balls B_r centered at the origin in terms of the surface measure σ on the boundary ∂B_r. Throughout this section $\Omega^\pm$ denotes the open sets of positive and negative values of h, $\Omega^\pm = \{h^\pm > 0\}$.

Lemma 4.3 ([3] Lemma 4.2)**.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial, $h(0) = 0$. For any $r > 0$,*

$$\omega_h(B_r) = \int_{\partial B_r \cap \Omega^+} \frac{\partial h^+}{\partial r} d\sigma = \int_{\partial B_r \cap \Omega^-} \frac{\partial h^-}{\partial r} d\sigma. \tag{4.6}$$

If h is homogeneous of degree k, then

$$\omega_h(B_r) = \frac{k}{2} r^{n+k-2} \|h\|_{L^1(S^{n-1})}. \tag{4.7}$$

Proof. For any harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree $d \geq 1$ with $h(0) = 0$, $B_r \cap \Omega^\pm$ is a non-empty set of locally finite perimeter. By the generalized Gauss-Green theorem,

$$\int_{\partial(B_r\cap\Omega^\pm)} \frac{\partial h^\pm}{\partial \nu^\pm} d\sigma = \int_{B_r \cap \partial\Omega^\pm} \Delta h^\pm = 0 \tag{4.8}$$

where $\nu^{\pm}$ denotes the unique outer unit normal defined at σ-a.e. $Q \in \partial(B_r \cap \Omega^{\pm})$. Thus, writing $\partial(B_r \cap \Omega^{\pm}) = (\partial B_r \cap \Omega^{\pm}) \cup (B_r \cap \partial\Omega^{\pm})$,

$$\int_{\partial B_r \cap \Omega^{\pm}} \frac{\partial h^{\pm}}{\partial r} d\sigma = -\int_{B_r \cap \partial\Omega^{\pm}} \frac{\partial h^{\pm}}{\partial \nu^{\pm}} d\sigma = \omega_h(B_r) \tag{4.9}$$

as desired.

Summing the two formulas in (4.6),

$$2\omega_h(B_r) = \int_{\partial B_r \cap \Omega^+} \frac{\partial h^+}{\partial r} d\sigma + \int_{\partial B_r \cap \Omega^-} \frac{\partial h^-}{\partial r} d\sigma. \tag{4.10}$$

If $h(r\theta) = r^k h(\theta)$, then $\partial_r h(r\theta) = kr^{k-1}h(\theta)$ and $r\theta \in \Omega^{\pm}$ if and only if $\theta \in \Omega^{\pm}$. Hence

$$\begin{aligned} 2\omega_h(B_r) &= \int_{\partial B_r \cap \Omega^+} kr^{k-1}h^+(\theta)d\sigma + \int_{\partial B_r \cap \Omega^-} kr^{k-1}h^-(\theta)d\sigma \\ &= \int_{\partial B_r} kr^{k-1}|h(\theta)|d\sigma = kr^{n+k-2}\int_{\partial B_1} |h(\theta)|d\sigma \end{aligned} \tag{4.11}$$

whenever h is homogeneous of degree k. □

A consequence of (4.7) is that the measures in $\mathcal{F}_k$ are uniformly doubling at the origin, i.e. for any $\omega_h \in \mathcal{F}_k$ and $r > 0$,

$$\frac{\omega_h(B_{2r})}{\omega_h(B_r)} = 2^{n+k-2} < \infty. \tag{4.12}$$

We now investigate the doubling properties of measures associated to arbitrary harmonic polynomials. The inequality for spherical harmonics in Corollary 3.21 is key.

Lemma 4.4 ([3] Lemma 4.3)**.** *Let $h : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial of degree $d \geq 1$ with $h(0) = 0$. There exists $r_1 = r_1(n, d, \zeta(h)) \geq 1$ such that for all $r > r_1$,*

$$\frac{l_{n,d}}{4} \cdot dr^{n+d-2}\|h_d\|_{L^\infty(S^{n-1})} \leq \omega_h(B_r) \leq \frac{3\sigma_{n-1}}{2} \cdot dr^{n+d-2}\|h_d\|_{L^\infty(S^{n-1})}. \tag{4.13}$$

Here $\zeta(h) = \zeta_d(h, 0, 1)$ and $r_1 = 1 + 12\sigma_{n-1}\zeta(h)/l_{n,d}$.

Proof. Without loss of generality assume that $M = \|h_d\|_{L^\infty(S^{n-1})} = \|h_d^+\|_{L^\infty(S^{n-1})}$; that is, the maximum of the homogeneous part h_d of h over S^{n-1} is obtained at a positive value.

Writing h in polar coordinates,

$$h(r\theta) = r^d h_d(\theta) + r^{d-1}h_{d-1}(\theta) + \cdots + rh_1(\theta), \tag{4.14}$$

$$\frac{\partial h}{\partial r}(r\theta) = dr^{d-1}h_d(\theta) + (d-1)r^{d-2}h_{d-1}(\theta) + \cdots + h_1(\theta). \tag{4.15}$$

Let $r > 1$. Then $\frac{1}{r} + \cdots + \left(\frac{1}{r}\right)^{d-1} \leq \sum_{i=1}^{\infty} \left(\frac{1}{r}\right)^i = \frac{1}{r-1}$ and with $\zeta(h)$ defined as above,

$$\left|\frac{r^{d-1}h_{d-1}(\theta) + \cdots + rh_1(\theta)}{r^d}\right| \leq M\zeta(h)\left(\frac{1}{r} + \cdots + \frac{1}{r^{d-1}}\right) \leq \frac{M\zeta(h)}{r-1}, \tag{4.16}$$

$$\left|\frac{(d-1)r^{d-2}h_{d-1}(\theta) + \cdots + h_1(\theta)}{r^{d-1}}\right| \leq dM\zeta(h)\left(\frac{1}{r} + \cdots + \frac{1}{r^{d-1}}\right) \leq \frac{dM\zeta(h)}{r-1}. \tag{4.17}$$

If $r\theta \in \partial B_r \cap \Omega^+$, then $h(r\theta) > 0$ and by (4.14) and (4.16),

$$h_d(\theta) > -\frac{r^{d-1}h_{d-1}(\theta) + \cdots + rh_1(\theta)}{r^d} \geq -\frac{M\zeta(h)}{r-1}. \tag{4.18}$$

Similarly, for all $r > 1$ and $\theta \in S^{n-1}$, by (4.15) and (4.17),

$$dr^{d-1}\left(h_d(\theta) - \frac{M\zeta(h)}{r-1}\right) \leq \frac{\partial h}{\partial r}(r\theta) \leq dr^{d-1}\left(h_d(\theta) + \frac{M\zeta(h)}{r-1}\right). \tag{4.19}$$

To estimate $\omega_h(B_r)$ for $r \gg 1$, we will combine (4.6), (4.18) and (4.19) with Corollary 3.21. By the latter, the set $\Gamma = \{\theta \in S^{n-1} : h_d(\theta) \geq M/2\}$ has surface measure $\sigma(\Gamma) \geq l_{n,d}$. Note that $r\Gamma \subset \partial B_r \cap \Omega^+$ provided $r > 1 + 2\zeta(h)$, since $h(r\theta) \geq r^d(h_d(\theta) - M\zeta(h)/(r-1))$, again by (4.14) and (4.16). Put $\Lambda_r = (\partial B_r \cap \Omega^+) \setminus r\Gamma$. Then, by (4.6) and (4.19),

$$\omega_h(B_r) \geq dr^{d-1}\int_{\partial B_r \cap \Omega^+}\left(h_d(\theta) - \frac{M\zeta(h)}{r-1}\right)d\sigma \tag{4.20}$$

$$\geq dr^{d-1}\int_{r\Gamma}\left(\frac{M}{2} - \frac{M\zeta(h)}{r-1}\right)d\sigma + dr^{d-1}\int_{\Lambda_r}\left(-\frac{M\zeta(h)}{r-1} - \frac{M\zeta(h)}{r-1}\right)d\sigma, \tag{4.21}$$

where $h_d(\theta) \geq M/2$ on Γ by definition and $h_d(\theta) > -M\zeta(h)/(r-1)$ for $r\theta \in \Lambda_r$ by (4.18). Since $\sigma(r\Gamma) \geq l_{n,d}r^{n-1}$ and $\sigma(\Lambda_r) \leq \sigma_{n-1}r^{n-1}$,

$$\omega_h(B_r) \geq dr^{d-1}M\left(\frac{1}{2} - \frac{\zeta(h)}{r-1}\right)l_{n,d}r^{n-1} + dr^{d-1}M\left(-\frac{2\zeta(h)}{r-1}\right)\sigma_{n-1}r^{n-1} \tag{4.22}$$

$$\geq dr^{n+d-2}M\left(\frac{l_{n,d}}{2} - \frac{3\sigma_{n-1}\zeta(h)}{r-1}\right). \tag{4.23}$$

Thus, if $r > 1+12\sigma_{n-1}\zeta(h)/l_{n,d}$, we obtain the lower bound $\omega_h(B_r) \geq (l_{n,d}/4)dr^{n+d-2}M$. A similar (and easier!) estimate using the upper bound in (4.19) shows if $r > 1+2\zeta(h)$ then $\omega_h(B_r) \leq (3\sigma_{n-1}/2)dr^{n+d-2}M$. Therefore, it suffices to take $r_1 = 1 + 12\sigma_{n-1}\zeta(h)/l_{n,d}$. □

As an immediate corollary of Lemma 4.4 we see that $\omega_h(B_r)$ is doubling as $r \to \infty$ with doubling constants depending only on n and d in the following sense.

Theorem 4.5 ([3] Theorem 4.4)**.** *There is a constant $C_{n,d} > 1$ such that for every $\tau > 1$ and every harmonic measure ω_h associated to a harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree d with $h(0) = 0$,*

$$\frac{\tau^{n+d-2}}{C_{n,d}} \leq \liminf_{r\to\infty} \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq \limsup_{r\to\infty} \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq C_{n,d}\tau^{n+d-2}. \tag{4.24}$$

Proof. By Lemma 4.4 there exists $r_1 \geq 1$ depending on ω_h such that for all $r > r_1$,

$$\frac{l_{n,d}}{6\sigma_{n-1}}\tau^{n+d-2} \leq \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq \frac{6\sigma_{n-1}}{l_{n,d}}\tau^{n+d-2}. \tag{4.25}$$

Thus, $C_{n,d} = 6\sigma_{n-1}/l_{n,d}$ suffices. □

While the top degree term of the polynomial h determines the harmonic measure $\omega_h(B_r)$ for large r, the non-zero term of lowest degree controls $\omega_h(B_r)$ on small radii.

Lemma 4.6 ([3] Lemma 4.5)**.** *Suppose that $h = h_d + h_{d-1} + \cdots + h_j$ is a harmonic polynomial with $1 \leq j \leq d$ and $h_j \neq 0$. There exists $r_2 = r_2(n, j, \zeta_*(h)) \leq 1/2$ such that for all $r < r_2$,*

$$\frac{l_{n,j}}{4} \cdot jr^{n+j-2}\|h_j\|_{L^\infty(S^{n-1})} \leq \omega_h(B_r) \leq \frac{3\sigma_{n-1}}{2} \cdot jr^{n+j-2}\|h_j\|_{L^\infty(S^{n-1})}. \tag{4.26}$$

Here $\zeta_*(h) = \zeta_j(h, 0, 1)$ and $r_2 = \min(1/2, l_{n,j}/72\sigma_{n-1}\zeta_*(h))$.

Proof. Without loss of generality assume that $M = \|h_j\|_{L^\infty(S^{n-1})} = \|h_j^+\|_{L^\infty(S^{n-1})}$; that is, the maximum of the homogeneous part h_j of h over S^{n-1} is obtained at a positive value.

Writing h in polar coordinates,

$$h(r\theta) = r^d h_d(\theta) + \cdots + r^{j+1} h_{j+1}(\theta) + r^j h_j(\theta), \tag{4.27}$$

$$\frac{\partial h}{\partial r}(r\theta) = dr^{d-1} h_d(\theta) + \cdots + (j+1) r^j h_{j+1}(\theta) + jr^{j-1} h_j(\theta). \tag{4.28}$$

Let $r \leq 1/2$. Then $r + \cdots + r^{d-j} \leq \sum_{i=1}^{\infty} r^i = \frac{r}{1-r} \leq 2r$ and with $\zeta_*(h)$ defined as above,

$$\left| \frac{r^d h_d(\theta) + \cdots + r^{j+1} h_{j+1}(\theta)}{r^j} \right| \leq M\zeta_*(h) \left(r^{d-j} + \cdots + r \right) \leq 2M\zeta_*(h) r. \tag{4.29}$$

Also, since $(j+i)/2j \leq i$ for all $i, j \geq 1$ and $\sum_{i=1}^{\infty} ir^i = \frac{r}{(1-r)^2} \leq 4r$,

$$\begin{aligned} \left| \frac{dr^{d-1} h_d(\theta) + \cdots + (j+1) r^j h_{j+1}(\theta)}{r^{j-1}} \right| &\leq M\zeta_*(h)(dr^{d-j} + \cdots + (j+1)r) \\ = 2jM\zeta_*(h) \left(\frac{d}{2j} r^{d-j} + \cdots + \frac{j+1}{2j} r \right) &\leq 2jM\zeta_*(h) \sum_{i=1}^{\infty} ir^i \leq 8jM\zeta_*(h) r. \end{aligned} \tag{4.30}$$

If $r\theta \in \partial B_r \cap \Omega^+$, then $h(r\theta) > 0$ and by (4.27) and (4.29),

$$h_j(\theta) > -\frac{r^d h_d(\theta) + \cdots + r^{j+1} h_{j+1}(\theta)}{r^j} \geq -2M\zeta_*(h) r. \tag{4.31}$$

Similarly, for all $r \leq 1/2$ and $\theta \in S^{n-1}$, by (4.28) and (4.30),

$$jr^{j-1} \left(h_j(\theta) - 8M\zeta_*(h) r \right) \leq \frac{\partial h}{\partial r}(r\theta) \leq jr^{j-1} \left(h_j(\theta) + 8M\zeta_*(h) r \right). \tag{4.32}$$

By Corollary 3.3, the set $\Gamma = \{\theta \in S^{n-1} : h_j(\theta) \geq M/2\}$ has surface measure $\sigma(\Gamma) \geq l_{n,j}$. Note $r\Gamma \subset \partial B_r \cap \Omega^+$ if $r < 1/4\zeta_*(h)$, since $h(r\theta) \geq r^j(h_j(\theta) - 2M\zeta_*(h) r)$, again by (4.27) and (4.29). Put $\Lambda_r = (\partial B_r \cap \Omega^+) \setminus r\Gamma$. Then, by (4.6) and (4.32),

$$\omega_h(B_r) \geq jr^{j-1} \int_{\partial B_r \cap \Omega^+} (h_j(\theta) - 8M\zeta_*(h) r) \, d\sigma \tag{4.33}$$

$$\geq jr^{j-1} M \int_{r\Gamma} \left(\frac{1}{2} - 8\zeta_*(h) r \right) d\sigma + jr^{j-1} M \int_{\Lambda_r} (-2\zeta_*(h) r - 8\zeta_*(h) r) \, d\sigma, \tag{4.34}$$

where $h_j(\theta) \geq M/2$ on Γ by definition and $h_j(\theta) > -2M\zeta_*(h) r$ for $r\theta \in \Lambda_r$ by (4.31). Since $\sigma(r\Gamma) \geq l_{n,j} r^{n-1}$ and $\sigma(\Lambda_r) \leq \sigma_{n-1} r^{n-1}$, if $r < 1/16\zeta_*(h)$ we obtain

$$\omega_h(B_r) \geq jr^{j-1} M \left(\frac{1}{2} - 8\zeta_*(h) r \right) l_{n,j} r^{n-1} + jr^{j-1} M \left(-10\zeta_*(h) r \right) \sigma_{n-1} r^{n-1} \tag{4.35}$$

$$\geq jr^{n+j-2} M \left(\frac{l_{n,j}}{2} - 18\sigma_{n-1} \zeta_*(h) r \right). \tag{4.36}$$

Thus, if $r < \min(1/2, l_{n,j}/72\sigma_{n-1}\zeta_*(h))$, we get the lower bound

$$\omega_h(B_r) \geq (l_{n,j}/4)jr^{n+j-2}M. \tag{4.37}$$

The estimate $\omega_h(B_r) \leq (3\sigma_{n-1}/2)jr^{n+j-2}M$ for all $r < \min(1/2, 1/16\zeta_*(h))$ is easier and follows from (4.6) and the upper bound in (4.32). Therefore, the estimates (4.26) for $\omega_h(B_r)$ hold for all $r < r_2$ with $r_2 = \min(1/2, l_{n,j}/72\sigma_{n-1}\zeta_*(h))$. □

Theorem 4.7 ([3] Theorem 4.6)**.** *There is a constant $c_{n,j} > 1$ such that for every $\tau > 1$ and every harmonic measure ω_h associated to a polynomial $h = h_d + h_{d-1} + \cdots + h_j$ with $1 \leq j \leq d$ and $h_j \neq 0$,*

$$\frac{\tau^{n+j-2}}{c_{n,j}} \leq \liminf_{r\to 0} \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq \limsup_{r\to 0} \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq c_{n,j}\tau^{n+j-2}. \tag{4.38}$$

Proof. By Lemma 4.6 there exists $r_2 \leq 1/2$ depending on ω_h such that whenever $\tau r < r_2$,

$$\frac{l_{n,j}}{6\sigma_{n-1}}\tau^{n+j-2} \leq \frac{\omega_h(B_{\tau r})}{\omega_h(B_r)} \leq \frac{6\sigma_{n-1}}{l_{n,j}}\tau^{n+j-2}. \tag{4.39}$$

Thus, $c_{n,j} = 6\sigma_{n-1}/l_{n,j}$ suffices. □

The next lemma generalizes Lemma 4.1 in [20]; notice that the assumption $\{h > 0\}$ and $\{h < 0\}$ are NTA domains has been removed.

Lemma 4.8 ([3] Lemma 4.7)**.** *Suppose $h : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial of degree $d \geq 1$ with $h(0) = 0$, and let ω_h be harmonic measure associated to h. There exists $\epsilon_0 > 0$ depending only on n, d and k such that if $d_r(\omega_h, \mathcal{F}_k) < \epsilon_0$ for all $r \geq r_0$ then $d = k$.*

Proof. Let $\tau > 1$ and choose $r \geq r_0$ such that $d_{\tau r}(\omega_h, \mathcal{F}_k) < \epsilon_0$. Then there exists $\psi \in \mathcal{F}_k$ such that $F_{\tau r}(\psi) = 1$ and

$$F_r\left(\frac{\omega_h}{F_{\tau r}(\omega_h)}, \psi\right) \leq F_{\tau r}\left(\frac{\omega_h}{F_{\tau r}(\omega_h)}, \psi\right) < \epsilon_0. \tag{4.40}$$

Hence, by the triangle inequality,

$$F_r(\psi) - \epsilon_0 < \frac{F_r(\omega_h)}{F_{\tau r}(\omega_h)} < F_r(\psi) + \epsilon_0. \tag{4.41}$$

Since ψ is associated to a homogeneous polynomial of degree k, say p, by Lemma 4.3,

$$F_r(\psi) = \int_0^r \psi(B_s)ds = \frac{k\|p\|_{L^1(S^{n-1})}}{2}\int_0^r s^{n+k-2}ds = \frac{k\|p\|_{L^1(S^{n-1})}}{2(n+k-1)}r^{n+k-1} \tag{4.42}$$

for all $r > 0$. In particular, $1 = F_{\tau r}(\psi) = \tau^{n+k-1}F_r(\psi)$. That is,

$$F_r(\psi) = \tau^{-n-k+1}. \tag{4.43}$$

Moreover, since $(r/2)\omega_h(B_{r/2}) \leq F_r(\omega_h) \leq r\omega_h(B_r)$ for all r, by Theorem 4.5,

$$\frac{1}{C_{n,d}}\left(\frac{1}{2\tau}\right)^{n+d-1} \leq \frac{1}{2\tau}\frac{\omega_h(B_{r/2})}{\omega_h(B_{\tau r})} \leq \frac{F_r(\omega_h)}{F_{\tau r}(\omega_h)} \leq \frac{2}{\tau}\frac{\omega_h(B_r)}{\omega_h(B_{\tau r/2})} \leq C_{n,d}\left(\frac{2}{\tau}\right)^{n+d-1} \tag{4.44}$$

for all $r > r_1(h)$. Setting $\widetilde{C} = C_{n,d}2^{n+d-1} > 1$,

$$\widetilde{C}^{-1}\tau^{-n-d+1} \leq \frac{F_r(\omega_h)}{F_{\tau r}(\omega_h)} \leq \widetilde{C}\tau^{-n-d+1} \tag{4.45}$$

for all $r > r_1$. Combining (4.41), (4.43) and (4.45) yields

$$\tau^{-n-k+1} - \epsilon_0 < \widetilde{C}\tau^{-n-d+1} \quad \text{and} \quad \widetilde{C}^{-1}\tau^{-n-d+1} < \tau^{-n-k+1} + \epsilon_0. \tag{4.46}$$

Equivalently,

$$\tau^{d-k}(1 - \tau^{n+k-1}\epsilon_0) < \widetilde{C} \quad \text{and} \quad \tau^{k-d}(1 + \tau^{n+k-1}\epsilon_0)^{-1} < \widetilde{C}. \tag{4.47}$$

Because $\widetilde{C}$ is independent of τ, we can set $\tau = 2\widetilde{C}$. Thus, for $(2\widetilde{C})^{n+k-1}\epsilon_0 = 1/2$,

$$\frac{1}{2}(2\widetilde{C})^{d-k} < \widetilde{C} \quad \text{and} \quad \frac{2}{3}(2\widetilde{C})^{k-d} < \widetilde{C}. \tag{4.48}$$

On a moment's reflection one sees (4.48) is impossible if $d \neq k$. (For example, if $d-k \geq 1$, then $\widetilde{C} = \frac{1}{2}(2\widetilde{C}) \leq \frac{1}{2}(2\widetilde{C})^{d-k} < \widetilde{C}$. If $k - d \geq 1$, then $\frac{4}{3}\widetilde{C} = \frac{2}{3}(2\widetilde{C}) \leq \frac{2}{3}(2\widetilde{C})^{k-d} < \widetilde{C}$.) Therefore, if $d_r(\omega_h, \mathcal{F}_k) < \epsilon_0 = \frac{1}{2}(2\widetilde{C})^{-n-k+1}$ for all $r \geq r_0$ then h has degree k. □

For emphasis let us remark again that ϵ_0 in Lemma 4.8 only depends on the dimension, the degree d of the polynomial h and the degree k of the "homogeneous cone" $\mathcal{F}_k$. Taking the minimum of finitely many ϵ_0 from Lemma 4.8 we obtain:

Corollary 4.9 ([3] Corollary 4.8). *There is $\epsilon_1 = \epsilon_1(n,d) > 0$ with the property if $\omega_h \in \mathcal{P}_d$ and $d_r(\omega_h, \mathcal{F}_k) < \epsilon_1$ for all $r \geq r_0$ with $1 \leq k \leq d$ then the degree of the polynomial associated to ω_h is k.*

Corollary 4.10 ([3] Corollary 4.9). *There is $\epsilon_2 = \epsilon_2(n,d) > 0$ with the property if $\omega_h \in \mathcal{P}_d$ and $d_r(\omega_h, \mathcal{F}_1) < \epsilon_2$ for all $r \geq r_0$ then $\omega_h \in \mathcal{F}_1$.*

In order to invoke Theorem 2.35 the cones studied must satisfy a compactness condition. Recall that the basis of a dilation invariant cone $\mathcal{M}$ is $\{\psi \in \mathcal{M} : F_1(\psi) = 1\}$.

Lemma 4.11 ([3] Lemma 4.10). *For each $k \geq 1$, $\mathcal{F}_k$ has a compact basis.*

Proof. First we claim there exists a constant $C = C(n,k) < \infty$ such that the coefficients of any polynomial associated to a harmonic measure in the basis of $\mathcal{F}_k$ are bounded by C. Let $\omega_h \in \mathcal{F}_k$ satisfying $F_1(\omega_h) = 1$ be associated to the homogeneous harmonic polynomial h of degree k. By (4.7) and the definition of F_1,

$$F_1(\omega_h) = \int_0^1 \omega_h(B_s)ds = \frac{k}{2(n+k-1)}\|h\|_{L^1(S^{n-1})}. \tag{4.49}$$

Since $F_1(\omega_h) = 1$, $\|h\|_{L^1(S^{n-1})} = 2(n+k-1)/k$. Hence, by Corollary 3.22,

$$\|h\|_{L^\infty(S^{n-1})} \leq B_{n,k}\|h\|_{L^1(S^{n-1})} = \frac{2B_{n,k}(n+k-1)}{k}. \tag{4.50}$$

If $h(X) = \sum_{|\alpha|=k} c_\alpha X^\alpha$ then $|c_\alpha| = |D^\alpha h(0)|/\alpha! \leq |D^\alpha h(0)|$ by Taylor's formula. Then the mean value property for $D^\alpha h$ and estimate (3.24) yield

$$|c_\alpha| \leq \fint_{S^{n-1}} |D^\alpha h(\theta)| d\sigma(\theta) \leq \sup_{\theta \in S^{n-1}} |D^\alpha h(\theta)| \leq (2^{n+2}nk)^k \|h\|_{L^\infty(S^{n-1})}. \tag{4.51}$$

Combining (4.50) and (4.51) shows that $|c_\alpha| \leq C(n,k)$ for every coefficient of h.

Now let $\omega^i \in \mathcal{F}_k$ be any sequence of measures such that $F_1(\omega^i) = 1$, and let h^i be the polynomial associated to ω^i. By the argument above, the coefficients of h^i are uniformly bounded. Hence from h^i we can extract a subsequence $h^{i_j} \to h^\infty$ uniformly on compact subsets of $\mathbb{R}^n$, where h^∞ is either identically zero or a homogeneous harmonic polynomial of degree k. (We will exclude the first possibility shortly). If $\varphi \in C_c^\infty(\mathbb{R}^n)$, then

$$\lim_{j\to\infty} \int \varphi d\omega^{i_j} = \lim_{j\to\infty} \int (h^{i_j})^+ \Delta\varphi = \int (h^\infty)^+ \Delta\varphi = \int \varphi d\omega_{h^\infty}. \tag{4.52}$$

Thus $\omega^{i_j} \rightharpoonup \omega^\infty = \omega_{h^\infty}$ and since $F_1(\omega^\infty) = \lim_{j\to\infty} F_1(\omega^{i_j}) = 1$, $h^\infty \not\equiv 0$. We have shown that for every sequence $\omega^i \in \mathcal{F}_k$ with $F_1(\omega^i) = 1$ there is a subsequence $\omega^{i_j} \rightharpoonup \omega^\infty \in \mathcal{F}_k$. Therefore, $\mathcal{F}_k$ has a compact basis. $\square$

We do not know if the cone $\mathcal{P}_d$ has a closed or compact basis for $d \geq 2$. To implement the method of Lemma 4.11 and show that $\mathcal{P}_d$ has a compact basis, one must find a way to control $\|h\|_{L^\infty(S^{n-1})}$ from the data $F_1(\omega_h) = 1$. On the other hand, to prove that $\mathcal{P}_d$ does not have a compact basis, by Proposition 2.37 it suffice to produce a sequence of measures $\omega_i \in \mathcal{P}_d$ and radii $r_i > 0$ such that $\sup_i \omega_i(B_{2r_i})/\omega_i(B_{r_i}) = \infty$. Since polynomial harmonic measures are doubling near infinity (Theorem 4.5) and doubling near zero (Theorem 4.7), candidate radii must be selected from an intermediate range depending on $\zeta(h)$ and $\zeta_*(h)$. The main challenge lies in estimating $\omega_h(B_r)$ on these middle scales. Since $\zeta(h)\zeta_*(h) \leq 1$ for every quadratic polynomial h, the final answer may depend on whether $d = 2$ or $d \geq 3$.

Chapter 5

HARMONIC MEASURE FROM TWO SIDES

In §5.1 we start by recalling the definition of NTA domains and two of the important features of their harmonic measures. First harmonic measure on an NTA domain is locally doubling. Second on an NTA domain there is a one-to-one correspondence between the geometric blow-ups of the domain and its boundary, the blow-ups of the Green functions and the tangent measures of harmonic measure. We discuss both one-sided and two-sided variants of NTA domains.

In §5.2 we return to the two-phase free boundary problem studied by Kenig and Toro [24] and reformulate their main results in terms of polynomial harmonic measures. They show that under certain conditions on the interior harmonic measure ω^+ and exterior harmonic measure ω^- on a two-sided NTA domain, $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d$ for some $d \geq 1$ depending only on the NTA constants of Ω. Then, using Chapter 4 results to check the criterion for connectedness of the cone of tangent measures (Theorem 2.35), we show that in fact $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ for some $1 \leq k \leq d$. Thus under the appropriate assumptions on ω^+ and ω^-, the boundary decomposes as $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d$ where $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{F}_k$ for all $Q \in \Gamma_k$.

In §5.3, we study the structure and size of the sets Γ_k. First we use Theorem 3.2 to show that the set of "flat points" Γ_1 is a relatively open subset of $\partial\Omega$. Second we use a general property of tangent measures (specifically almost everywhere translation invariance) to show that the set of "singularities" $\Gamma_2 \cup \cdots \cup \Gamma_d$ has harmonic measure zero.

5.1 Harmonic Measure and Blow-ups on NTA Domains

Jerison and Kenig introduced non-tangentially accessible domains in $\mathbb{R}^n$ as a natural class of domains on which Fatou type convergence theorems hold for harmonic functions [15]. A bounded simply connected domain $\Omega \subset \mathbb{R}^2$ is an NTA domain if and only if Ω is a quasidisk (the image of the unit disk under a global quasiconformal mapping of the plane). While every quasiball in $\mathbb{R}^n$, $n \geq 3$, is also a bounded NTA domain, there exist bounded NTA domains homeomorphic to a ball in $\mathbb{R}^n$ which are not quasiballs. The reader may consult [15] for more information. Also see [21] where Kenig and Toro demonstrate that a domain in $\mathbb{R}^n$ whose boundary is δ-Reifenberg flat is an NTA domain provided that $\delta < \delta_n$ is sufficiently small.

The definition of NTA domains is based on two geometric conditions.

Definition 5.1. An open set $\Omega \subset \mathbb{R}^n$ satisfies the *corkscrew condition* with constants $M > 1$ and $R > 0$ provided that for every $Q \in \partial\Omega$ and $0 < r < R$ there exists a *non-tangential point* $A = A(Q,r) \in \Omega \cap B(Q,r)$ such that $\operatorname{dist}(A, \partial\Omega) > M^{-1}r$.

For $X_1, X_2 \in \Omega$ a *Harnack chain* from X_1 to X_2 is a sequence of closed balls inside Ω such that the first ball contains X_1, the last contains X_2, and consecutive balls intersect.

Definition 5.2. A domain $\Omega \subset \mathbb{R}^n$ satisfies the *Harnack chain condition* with constants $M > 1$ and $R > 0$ if for every $Q \in \partial\Omega$ and $0 < r < R$ when $X_1, X_2 \in \Omega \cap B(Q,r)$ satisfy

$$\min_{j=1,2} \operatorname{dist}(X_j, \partial\Omega) > \varepsilon \quad \text{and} \quad |X_1 - X_2| < 2^k \varepsilon \tag{5.1}$$

then there is a Harnack chain from X_1 to X_2 of length Mk such that the diameter of each ball is bounded below by $M^{-1} \min_{j=1,2} \operatorname{dist}(X_j, \partial\Omega)$.

Definition 5.3. A domain $\Omega \subset \mathbb{R}^n$ is *non-tangentially accessible* or *NTA* if there exist $M > 1$ and $R > 0$ such that (i) Ω satisfies the corkscrew and Harnack chain conditions, (ii) $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew condition. If $\partial\Omega$ is unbounded then we require $R = \infty$.

The exterior corkscrew condition guarantees that every NTA domain is regular for the Dirichlet problem. This means that the Perron solution of the Dirichlet problem coincides with the classical solution of the Dirichlet problem, in the sense that $Hf \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfies (1.2) for every $f \in C_c(\partial\Omega)$. We also want to emphasize that Definition 5.3 of a NTA domain only requires the Harnack chain condition on the interior of the domain and does not require the exterior of the domain be connected.

Definition 5.4. A domain $\Omega \subset \mathbb{R}^n$ is *two-sided non-tangentially accessible* or *2-sided NTA* if $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ are both NTA with the same constants; i.e., there exists $M > 1$ and $R > 0$ such that $\Omega^\pm$ satisfy the corkscrew and Harnack chain conditions. When $\partial\Omega$ is unbounded, we still require $R = \infty$.

To distinguish between NTA domains and 2-sided NTA domains, we may call the former 1-sided NTA domains. Note that both the interior and the exterior of a 2-sided NTA domain are 1-sided NTA domains.

We adopt the following conventions. If $\Omega \subset \mathbb{R}^n$ is a bounded 1-sided NTA domain, then ω $(= \omega_\Omega^{X_0})$ denotes harmonic measure of Ω with respect to some fixed pole $X_0 \in \Omega$ (in which case we say ω has finite pole). If Ω is unbounded, ω may denote either harmonic measure with a finite pole or with pole at infinity (see Lemma 5.6). If $\Omega \subset \mathbb{R}^n$ is a 2-sided NTA domain, then we write ω^+ to denote harmonic measure on the interior $\Omega^+ = \Omega$ and write ω^- to denote harmonic measure on the exterior $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$. If Ω is unbounded, we again allow ω^+ and ω^- to have finite poles or poles at infinity.

Harmonic measure on NTA domains is locally doubling [15]. While Jerison and Kenig only considered bounded NTA domains, their proof is local in nature and extends to the unbounded case without modification; c.f., [22].

Lemma 5.5 ([15] Lemmas 4.8, 4.11)**.** *For each $n \geq 2$ and $M > 1$ there exists a constant $C = C(n, M) > 1$ such that for every NTA domain $\Omega \subset \mathbb{R}^n$ with constants M and $R > 0$: if $Q \in \partial\Omega$, $0 < 2r < R$ and $X \in \Omega \setminus B(Q, 2Mr)$ then $\omega_\Omega^X(B(Q, 2s)) \leq C\omega_\Omega^X(B(Q, s))$ for all $0 < s < r$.*

On an unbounded NTA domain there is a related doubling measure called harmonic measure with pole at infinity, which is obtained as the weak limit of harmonic measures $\omega_\Omega^{X_i}$ (properly rescaled) as $X_i \to \infty$.

Lemma 5.6 ([22] Lemma 3.7, Corollary 3.2)**.** *Let $\Omega \subset \mathbb{R}^n$ be an unbounded NTA domain. There exists a doubling Radon measure ω_Ω^∞ supported on $\partial\Omega$ satisfying*

$$\int_{\partial\Omega} \varphi \, d\omega_\Omega^\infty = \int_\Omega u\Delta\varphi \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n) \tag{5.2}$$

where

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

The measure ω_Ω^∞ and Green function u are unique up to multiplication by a positive scalar. We call ω_Ω^∞ a harmonic measure of Ω with pole at infinity.

When a result about harmonic measure of a domain Ω is independent of the choice of pole, we denote the measure by ω without any superscript. This means that when Ω is unbounded we allow ω to have a finite pole or pole at infinity.

Lemma 5.7. *If $\Omega \subset \mathbb{R}^n$ is NTA and $Q \in \partial\Omega$, then $\operatorname{Tan}(\omega, Q)$ has a compact basis.*

Proof. At any point in the support, the tangent measures of an asymptotically doubling measure has a compact basis by Corollary 2.38. This is true on an NTA domain by Lemma 5.5 when ω has a finite pole and by Lemma 5.6 when ω has pole at infinity. □

On an NTA domain there is exists a one-to-one correspondence between the tangent measures of harmonic measure and geometric blow-ups of the domain and boundary [22]. Let $\Omega \subset \mathbb{R}^n$ be a NTA domain, let $Q \in \partial\Omega$ and let $r_i \downarrow 0$. For each i, zoom in on the domain, the boundary and the harmonic measure at Q and scale r_i:

$$\Omega_i = \frac{\Omega - Q}{r_i}, \quad \partial\Omega_i = \frac{\partial\Omega - Q}{r_i}, \quad \omega_i = \frac{T_{Q,r_i}[\omega]}{\omega(B(Q,r_i))}. \tag{5.4}$$

Theorem 5.8 ([22] Lemma 3.8)**.** *Let $\Omega \subset \mathbb{R}^n$ be a NTA domain, let $Q \in \partial\Omega$ and let $r_i \downarrow 0$. Define Ω_i, $\partial\Omega_i$ and ω_i by (5.4). There exists a subsequence of r_i (relabel it) and an unbounded NTA domain $\Omega_\infty \subset \mathbb{R}^n$ such that*

$$\Omega_i \to \Omega_\infty \quad \textit{in Hausdorff distance sense uniformly on compact sets}, \tag{5.5}$$

$$\partial\Omega_i \to \partial\Omega_\infty \quad \textit{in Hausdorff distance sense uniformly on compact sets}. \tag{5.6}$$

Moreover,

$$\omega_i \rightharpoonup \omega_\infty \tag{5.7}$$

where ω_∞ is harmonic measure for Ω_∞ with pole at infinity.

Remark 5.9. The measure ω_∞ in Theorem 5.8 obtained as a weak limit of the blow-ups $\omega(B(Q, r_i))^{-1} T_{Q,r_i}[\omega]$ is a tangent measure of ω at Q. In fact, up to scaling by a constant, every tangent measure of ω at Q has this form since ω is doubling; c.f. [29] Remark 14.4. Hence, since the blow-ups Ω_i of the domain Ω do not depend on the pole of harmonic measure, the cone of tangent measures $\operatorname{Tan}(\omega, Q)$ is also independent of the pole of ω.

There is a version of the blow-up procedure for 2-sided NTA domains as well [24]. We state the version where the fixed base point Q is replaced by a sequence of boundary points $Q_i \to Q$. Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain. Choose $(Q_i)_{i=1}^\infty \subset \partial\Omega$ such that $Q_i \to Q \in \partial\Omega$. Also choose $r_i \downarrow 0$. Let $u^\pm$ be the Green function for $\Omega^\pm$ with the same pole as the harmonic measure $\omega^\pm$. We zoom in on the interior and exterior domains, boundary, harmonic measures and Green functions at Q along scales r_i:

$$\begin{aligned} \Omega_i^\pm &= \frac{\Omega^\pm - Q_i}{r_i}, & \partial\Omega_i &= \frac{\partial\Omega - Q_i}{r_i}, \\ \omega_i^\pm &= \frac{T_{Q_i,r_i}[\omega^\pm]}{\omega^\pm(B(Q_i, r_i))}, & u_i^\pm &= \frac{u^\pm \circ T_{Q_i,r_i}^{-1}}{\omega^\pm(B(Q_i, r_i))} r_i^{n-2}. \end{aligned} \tag{5.8}$$

Theorem 5.10 ([24] Theorem 4.2)**.** *Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain. Given the sequences $Q_i \to Q \in \partial\Omega$ and $r_i \downarrow 0$, define the sets $\Omega_i^\pm$ and $\partial\Omega_i$, the measures $\omega_i^\pm$ and the* *functions $u_i^\pm$ by (5.8). There exists a subsequence of r_i (which we relabel) and an unbounded 2-sided NTA domain $\Omega_\infty \subset \mathbb{R}^n$ such that*

$$\Omega_i^\pm \to \Omega_\infty^\pm \quad \text{in Hausdorff distance uniformly on compact sets,} \tag{5.9}$$

$$\partial\Omega_i \to \partial\Omega_\infty \quad \text{in Hausdorff distance uniformly on compact sets.} \tag{5.10}$$

Moreover,

$$\omega_i^\pm \rightharpoonup \omega_\infty^\pm, \tag{5.11}$$

$$u_i^\pm \to u_\infty^\pm \quad \text{uniformly on compact sets} \tag{5.12}$$

where $\omega_\infty^\pm$ is harmonic measure with pole at infinity for $\Omega^\pm$ and Green function $u_\infty^\pm$.

Remark 5.11. The measures $\omega_\infty^\pm$ in Theorem 5.10 obtained as a weak limit of blow-ups $\omega^\pm(B(Q_i,r_i))^{-1}T_{Q_i,r_i}[\omega]$ are only *pseudo*tangent measures of $\omega^\pm$ at Q. If the base points $Q_i = Q$ remain centered for all $i \geq 1$, then $\omega_\infty^\pm$ are tangent measures. Because the measures $\omega^\pm$ are locally doubling all pseudotangent measures and all tangent measures of $\omega^\pm$ at Q (up to scaling by a constant) are obtained in this fashion.

5.2 Polynomial Harmonic Measures as Tangent Measures

Polynomial harmonic measures appear as tangent measures on 2-sided NTA domains which possess mutually absolutely continuous interior and exterior harmonic measures. To state the precise requirement, we require two definitions.

Definition 5.12. Let $\Omega \subset \mathbb{R}^n$ be a NTA domain. We say that $f \in L^2_{\mathrm{loc}}(d\omega)$ has *bounded mean oscillation* with respect to the harmonic measure ω and write $f \in \mathrm{BMO}(d\omega)$ if

$$\sup_{r>0}\sup_{Q\in\partial\Omega}\left(⨍_{B(Q,r)} |f - f_{Q,r}|^2 d\omega\right)^{1/2} < \infty \tag{5.13}$$

where $f_{Q,r} = ⨍_{B(Q,r)} f\,d\omega$ (the average of f over the ball).

Definition 5.13. Let $\Omega \subset \mathbb{R}^n$ be a NTA domain. Let $\mathrm{VMO}(d\omega)$ denote the closure of the set of bounded uniformly continuous functions on $\partial\Omega$ in $\mathrm{BMO}(d\omega)$. If $f \in \mathrm{VMO}(d\omega)$ we say f has *vanishing mean oscillation* with respect to the harmonic measure ω.

The content of the following theorem is due to Kenig and Toro [24]. It was first stated in this form in Badger [3].

Theorem 5.14 ([24] Theorem 4.4, [3] Theorem 6.5)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a 2-sided NTA domain. If* $\omega^+ \ll \omega^- \ll \omega^+$ *and* $f = d\omega^-/d\omega^+$ *satisfies* $\log f \in \mathrm{VMO}(d\omega^+)$*, then there exists* $d \geq 1$ *depending only on* n *and the NTA constants of* Ω *such that for all* $Q \in \partial\Omega$*,*

$$\Psi\text{–}\mathrm{Tan}(\omega^+, Q) = \Psi\text{–}\mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d \tag{5.14}$$

and

$$\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d. \tag{5.15}$$

Proof. Under the same hypothesis, Theorem 4.4 in Kenig and Toro [24] concludes (using the notation of Theorem 5.10 above) that $\omega^+_\infty = \omega^-_\infty$ and $u = u^+_\infty - u^-_\infty$ is a harmonic polynomial (for every choice of base points $Q_i \to Q$ and every choice of scales $r_i \to 0$). The proof that u is a polynomial shows there exists $d \geq 1$ determined by n and the NTA constants of Ω such that u has degree at most d. Thus, by the correspondence between pseudotangent measures and blow-ups of Green functions (see Remark 5.11), we conclude that $\Psi\text{–}\mathrm{Tan}(\omega^+, Q) = \Psi\text{–}\mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d$, the cone of polynomial harmonic measures of degree at most d. Restricting to the constant sequences $Q_i = Q$ for all $i \geq 1$, we conclude that $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d$, as well. □

Our next objective is to exhibit a "self-improving" property of the tangent measures $\mathrm{Tan}(\omega, Q)$ of harmonic measure ω at a point Q in the boundary of an NTA domain Ω. Recall $\mathrm{Tan}(\omega, Q)$ is independent of the choice of pole for ω (see Remark 5.9). When the domain is unbounded, we allow ω to have a finite pole or pole at infinity.

Theorem 5.15 ([3] Theorem 1.1)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be an NTA domain. If* $Q \in \partial\Omega$ *and* $\mathrm{Tan}(\omega, Q) \subset \mathcal{P}_d$*, then* $\mathrm{Tan}(\omega, Q) \subset \mathcal{F}_k$ *for some* $1 \leq k \leq d$*.*

We need a lemma to identify the degree k of the cone $\mathcal{F}_k$ appearing in Theorem 5.15.

Lemma 5.16 ([3] Lemma 5.9). *Assume that* $\Omega \subset \mathbb{R}^n$ *is NTA,* $Q \in \partial\Omega$ *and* $\mathrm{Tan}(\omega, Q) \subset \mathcal{P}_d$. *If* k *is the minimum degree such that* $\mathcal{P}_k \cap \mathrm{Tan}(\omega, Q) \neq \emptyset$*, then* $\mathcal{P}_k \cap \mathrm{Tan}(\omega, Q) \subset \mathcal{F}_k$.

Proof. If $k = 1$, then $\mathcal{P}_1 = \mathcal{F}_1$. If $k \geq 2$, suppose for contradiction that there exists $\nu_h \in \mathrm{Tan}(\omega, Q)$ associated to a nonhomogeneous harmonic polynomial h of degree k, say $h = h_k^{(0)} + h_{k-1}^{(0)} + \cdots + h_j^{(0)}$ with $j < k$ and $h_j \neq 0$. Note that by Theorem 5.8 (applied to Ω and ω) either $\{x \in \mathbb{R}^n : h(x) > 0\}$ or $\{x \in \mathbb{R}^n : h(x) < 0\}$ is an unbounded NTA domain for which ν_h is a harmonic measure with pole at infinity. Without loss of generality, assume $U = \{x \in \mathbb{R}^n : h(x) > 0\}$ is an unbounded NTA domain for which ν_h is a harmonic measure with pole at infinity. Choose a sequence $r_i \downarrow 0$. By Theorem 5.8 (now applied to U and ν_h), there exists a subsequence of r_i (which we relabel) and there exists an unbounded NTA domain U_∞ such that

$$U_i = \frac{U}{r_i} \to U_\infty \quad \text{and} \quad \partial U_i = \frac{\partial U}{r_i} \to \partial U_\infty \tag{5.16}$$

in the sense of Hausdorff distance uniformly on compact sets and

$$\nu_i = \frac{T_{0,r_i}[\nu_h]}{\nu_h(B_{r_i})} \rightharpoonup \nu_\infty \tag{5.17}$$

where ν_∞ is harmonic measure for U_∞ with pole at infinity. Since $\nu_h \in \mathrm{Tan}(\omega, Q)$ and $\nu_\infty \in \mathrm{Tan}(\nu_h, 0)$, we get $\nu_\infty \in \mathrm{Tan}(\omega, Q)$ by Lemma 2.28. Furthermore, $\partial U_\infty = \Sigma_{h_j^{(0)}}$ (the zero set of $h_j^{(0)}$) by Lemma 3.26. Thus $\nu_\infty \in \mathrm{Tan}(\omega, Q) \cap \mathcal{F}_j$. This contradicts the minimality of k. Therefore, every blow up of ω at Q of minimum degree is homogeneous: $\mathcal{P}_k \cap \mathrm{Tan}(\omega, Q) \subset \mathcal{F}_k$, as desired. □

Proof of Theorem 5.15. Let $k = \min\{j : \mathcal{P}_j \cap \mathrm{Tan}(\omega, Q) \neq \emptyset\} \leq d$ and set

$$\mathcal{F} = \mathcal{F}_k, \quad \mathcal{M} = \mathrm{Tan}(\omega, Q) \cup \mathcal{F}_k. \tag{5.18}$$

Then $\mathcal{F} \subset \mathcal{M}$ and both dilation invariant cones have a compact basis by Lemma 4.11 and Lemma 5.7. Since $\mathcal{M} \subset \mathcal{P}_d$, Corollary 4.9 and Lemma 5.16 together imply that there exists an $\epsilon_1 > 0$ such that for all $\mu \in \mathcal{M}$ if $d_r(\mu, \mathcal{F}_k) < \epsilon_1$ for all $r \geq r_0$ then $\mu \in \mathcal{F}_k$.

By Theorem 2.35 (the connectedness of tangent measures), since $\mathrm{Tan}(\omega, Q) \subset \mathcal{M}$ and $\mathrm{Tan}(\omega, Q) \cap \mathcal{F}_k \neq \emptyset$, we conclude $\mathrm{Tan}(\omega, Q) \subset \mathcal{F}_k$. □

Applied to Theorem 5.14, the self-improving property of tangent measures established in Theorem 5.15 immediately yields:

Corollary 5.17 ([3] Corollary 6.6)**.** *Let $\Omega \subset \mathbb{R}^n$ be 2-sided NTA. If $\omega^+ \ll \omega^- \ll \omega^+$ and $f = d\omega^-/d\omega^+$ satisfies $\log f \in \mathrm{VMO}(d\omega^+)$, then there exists $d \geq 1$ depending only on n and the NTA constants of Ω and there exists pairwise disjoint sets $\Gamma_1, \ldots, \Gamma_d$ such that*

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d, \tag{5.19}$$

where $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{F}_k$ for all for all $Q \in \Gamma_k$, for all $1 \leq k \leq d$.

Remark 5.18. One can also apply Theorem 5.15 to tangent measures on 2-sided domains without any assumptions on the Radon-Nikodym derivative $d\omega^-/d\omega^+$, as follows.

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary 2-sided NTA domain. We recall the definition of the set $\Gamma \subset \partial\Omega$ from [20]. By the differentiation theory of Radon measures,

$$h(Q) = \lim_{r \downarrow 0} \frac{\omega^-(B(Q,r))}{\omega^+(B(Q,r))} \in [0, \infty] \tag{5.20}$$

exists for $\omega^\pm$ almost every $Q \in \partial\Omega$. Define

$$\Lambda = \{Q \in \partial\Omega : h(Q) \text{ exists, } 0 < h(Q) < \infty\}. \tag{5.21}$$

It is easily seen that $\omega^+ \ll \omega^- \ll \omega^+$ on Λ and $\omega^+ \perp \omega^-$ on $\partial\Omega \setminus \Lambda$. (The paper [20] uses the notation 'Λ_1' for Λ. It also specifies sets Λ_2, Λ_3 and Λ_4, which we do not require here.) To define Γ we restrict our attention to density points of Λ and h:

$$\Gamma = \{Q \in \Lambda : Q \text{ is a density point of } \Lambda \text{ and a Lebesgue point of } h \text{ w.r.t. } \omega^+\}. \tag{5.22}$$

Then Γ agrees with Λ up to a set of $\omega^\pm$ measure zero, and any subset $A \subset \partial\Omega$ for which $\omega^+ \llcorner A \ll \omega^- \llcorner A \ll \omega^+ \llcorner A$ can be decomposed as $A = B \cup N$ where $\omega^\pm(N) = 0$ and $B \subset \Gamma$. Thus, up to a set of $\omega^\pm$ measure zero, the set Γ is the maximal "mutually absolutely continuous piece" of $\partial\Omega$. By Theorems 3.3 and 3.4 in [20] (analogously to Theorems 5.10 and 5.14), there exists $d \geq 1$ such that $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{P}_d$ for all $Q \in \Gamma$. Hence, by Theorem 5.15,

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_d, \tag{5.23}$$

where for each $Q \in \Gamma_k$, $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{F}_k$.

In particular, if $\Omega \subset \mathbb{R}^n$ is a 2-sided NTA domain and $\omega^+ \ll \omega^- \ll \omega^+$, then

$$\partial\Omega = \Gamma \cup N = \Gamma_1 \cup \cdots \cup \Gamma_d \cup N \tag{5.24}$$

where $\omega^\pm(N) = 0$ and $\mathrm{Tan}(\omega^+, Q) = \mathrm{Tan}(\omega^-, Q) \subset \mathcal{F}_k$ for each $Q \in \Gamma_k$.

5.3 Structure and Size of the Free Boundary

Next we examine properties of the sets $\Gamma_k \subset \partial\Omega$, $1 \leq k \leq d$, appearing in Corollary 5.17. First we classify the blow-ups of the boundary. At each point $Q \in \Gamma_k$, every blow-up of $\partial\Omega$ is the zero set Σ_h of a homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree k with the additional property that $\mathbb{R}^n \setminus \Sigma_h$ has exactly two connected components.

Theorem 5.19 ([3] Theorem 1.3)**.** *Let Ω be as in Corollary 5.17. Every blow-up of $\partial\Omega$ centered at $Q \in \Gamma_k$ is the zero set of a homogeneous harmonic polynomial of degree k separating $\mathbb{R}^n$ into two components.*

Proof. Let $Q \in \Gamma_k$ and suppose that $r_i^{-1}(\partial\Omega - Q)$ has a limit in the Hausdorff distance for some $r_i \downarrow 0$. By Theorem 5.10, there is a subsequence of r_i (which we relabel) such that $\omega^+(B(Q, r_i))^{-1} T_{Q,r_i}[\omega^+] \rightharpoonup \omega^+_\infty \in \mathrm{Tan}(\omega^+, Q)$ and

$$\frac{\partial\Omega - Q}{r_i} \to \mathrm{spt}\,\omega^+_\infty \quad \text{in Hausdorff distance uniformly on compact sets.} \tag{5.25}$$

Theorem 5.10 also concludes $\mathbb{R}^n \setminus \mathrm{spt}\,\omega^+_\infty$ is a 2-sided NTA domain (union its exterior). Since $Q \in \Gamma_k$, $\omega^+_\infty \in \mathrm{Tan}(\omega^+, Q) \subset \mathcal{F}_k$. Hence there exists a homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ of degree k such that $\mathrm{spt}\,\omega^+_\infty = \Sigma_h$, the zero set of h. Moreover, $\mathbb{R}^n \setminus \Sigma_h = \Omega^+_h \cup \Omega^-_h$ has exactly two connected components, because $\mathbb{R}^n \setminus \Sigma_h$ is a 2-sided NTA domain (union its exterior). □

The set of "flat points" Γ_1 where $\partial\Omega$ blows-up to a hyperplane is a relatively open subset of the boundary with full harmonic measure (see Theorem 5.23, Corollary 5.27). To show that Γ_1 is relatively open, we first fix notation for approximation by zero sets of harmonic polynomials. When $d = 1$, $\theta^d_A(x, r)$ coincides with $\theta_A(x, r)$ (see Definition 2.8).

Definition 5.20. Let $A \subset \mathbb{R}^n$ be a nonempty set and let $x \in A$. For all $d \geq 1$ and $r > 0$, define

$$\theta^d_A(x, r) = \frac{1}{r} \inf_V \mathrm{HD}(A \cap B(x, r), (x + V) \cap B(x, r)) \tag{5.26}$$

where V ranges over the zero sets of harmonic polynomials $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(0) = 0$ and $1 \leq \deg h \leq d$.

Lemma 5.21. *Let Ω be as in Corollary 5.17. Then $\lim_{r\downarrow 0} \theta^d_{\partial\Omega}(Q, r) = 0$ uniformly on compact subsets of $\partial\Omega$.*

Proof. Fix a compact set $K \subset \partial\Omega$, and suppose for contradiction that $\theta^d_{\partial\Omega}(Q, r)$ does not vanish uniformly on K. Then there exist $\varepsilon > 0$ and sequences $Q_i \in K$ and $r_i \downarrow 0$ so that

$$\theta^d_{\partial\Omega}(Q_i, r_i) \geq \varepsilon \quad \text{for all } i \geq 1. \tag{5.27}$$

Passing to a subsequence, we may assume that $Q_i \to Q \in K$ (since K is compact). Then Theorem 4.4 in [24] (also see Theorem 5.14 above) yields a further subsequence $(Q_{ij}, r_{ij})_{j=1}^\infty$ of $(Q_i, r_i)_{i=1}^\infty$ such that $\lim_{j\to\infty} \theta^d_{\partial\Omega}(Q_{ij}, r_{ij}) = 0$. This contradicts (5.27). Therefore, our supposition was false and $\lim_{r\downarrow 0} \theta^d_{\partial\Omega}(Q, r) = 0$ uniformly on K. □

If a set is uniformly close to the zero set of a harmonic polynomial on all small scales, then flatness at one scale automatically controls flatness on smaller scales.

Lemma 5.22. *For all $n \geq 2$, $d \geq 1$ and $\delta > 0$, there exist $\varepsilon = \varepsilon(n, d, \delta) > 0$ and $\eta = \eta(n, d, \delta) > 0$ with the following property. Let $A \subset \mathbb{R}^n$, $x \in A$, $r > 0$ and assume that*

$$\sup_{0<r'\leq r} \theta^d_A(x, r') < \varepsilon. \tag{5.28}$$

If $\theta^1_A(x, r) < \eta$, then $\sup_{0<r'\leq r} \theta^1_A(x, r') < \delta$.

Proof. Let $\delta > 0$ be given and fix parameters $\varepsilon > 0$, $\sigma > 0$, and $\tau > 0$ to be chosen later. Assume that $A \subset \mathbb{R}^n$ is a non-empty set which satisfies (5.28) for some $x \in A$ and $r > 0$. Also assume that $\theta^1_A(x,r) < \tau$. Then by definition there exists a hyperplane $L \in G(n, n-1)$ such that

$$\mathrm{HD}(A \cap B(x,r), (x+L) \cap B(x,r)) < \tau r. \tag{5.29}$$

On the other hand, since $\theta^d_A(x,r) < \varepsilon$, there exists a harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(0) = 0$ and $1 \leq \deg h \leq d$ for which

$$\mathrm{HD}(A \cap B(x,r), (x+\Sigma_h) \cap B(x,r)) < \varepsilon r. \tag{5.30}$$

Combining (5.29) and (5.30), we have

$$\mathrm{HD}(\Sigma_h \cap B_r, L \cap B_r) < (\tau + \varepsilon) r. \tag{5.31}$$

Set $\delta_* = \min\{\delta_{n,2}, \dots, \delta_{n,d}\}$, where $\delta_{n,2}, \dots, \delta_{n,d}$ denote the constants from Theorem 3.2, and assign $s = \sigma(\sqrt{5}(d-1)/\delta_*)^{-1}$. Assume that $\tau + \varepsilon < \delta_*$. By Theorem 3.2 and (5.31), $\theta^1_{\Sigma_h}(0, 2sr) < 2\sigma$. Hence there exists $P \in G(n, n-1)$ such that

$$\mathrm{HD}(\Sigma_h \cap B_{2sr}, P \cap B_{2sr}) < 2\sigma(2sr) = 4\sigma sr. \tag{5.32}$$

We will use (5.30) and (5.32) to estimate $\mathrm{HD}(A \cap B(x,sr), (x+P) \cap B(x,sr))$.

First suppose that $x' \in A \cap B(x,sr)$. By (5.30), $\mathrm{dist}(x', (x+\Sigma_h) \cap B(x,r)) < \varepsilon r$. Hence there exists $y \in \Sigma_h$ such that $|x' - x - y| < \varepsilon r$. We now specify that $\varepsilon \leq s \approx \sigma$. Then $y \in \Sigma_h \cap B_{sr+\varepsilon r} \subset \Sigma_h \cap B_{2sr}$. Hence by (5.32), $\mathrm{dist}(y, P \cap B_{2sr}) < 4\sigma sr$. Choose $p \in P \cap B_{2sr}$ such that $|y - p| < 4\sigma sr$. In fact, since $y \in B_{sr+\varepsilon r}$, we know $p \in B_{sr+\varepsilon r + 4\sigma sr}$. Since P is a plane through the origin, we can find a second point $p' \in P \cap B_{sr}$ such that $|p' - p| \leq \varepsilon r + 4\sigma sr$. Thus $x + p' \in (x+P) \cap B(x,sr)$ and

$$|x' - x - p'| \leq |x' - x - y| + |y - p| + |p - p'| < 2\varepsilon r + 8\sigma sr. \tag{5.33}$$

We conclude that

$$\mathrm{dist}(x', (x+P) \cap B(x,sr)) < 2\varepsilon r + 8\sigma sr \quad \text{for all } x' \in A \cap B(x,sr). \tag{5.34}$$

Next suppose that $x+p \in (x+P)\cap B(x,sr)$. Since P is a plane, we can select a second point $x+p' \in (x+P)\cap B(x, sr-\varepsilon r - 4\sigma sr)$ such that $|p'-p| \le \varepsilon r + 4\sigma sr$. By (5.32) there exists $x+y \in (x+\Sigma_h)\cap B(x,2sr)$ such that $|p'-y| < 4\sigma sr$. In fact, since $p \in B_{sr-\varepsilon r-4\sigma sr}$, we get $y \in B_{sr-\varepsilon r}$. By (5.30) there exists $x' \in A\cap B(x,r)$ with $|x+y-x'| < \varepsilon r$. But since $y \in B_{sr-\varepsilon r}$, we know $x' \in A \cap B(x,sr)$ and

$$|x+p-x'| \le |p-p'| + |p'-y| + |x+y-x'| < 2\varepsilon r + 8\sigma sr. \tag{5.35}$$

Thus

$$\operatorname{dist}(x+p, A\cap B(x,sr)) < 2\varepsilon r + 8\sigma sr \quad \text{for all } x+p \in (x+P)\cap B(x,sr) \tag{5.36}$$

Combining (5.34) and (5.36) yields

$$\theta^1_A(x,sr) < 4\varepsilon/s + 16\sigma \tag{5.37}$$

provided that $\tau + \varepsilon < \delta_*$ and $\varepsilon \le s$.

We are ready to choose parameters. Set $\tau = \min(\delta,\delta_*)/2$, put $\sigma = \tau/32$ (note $s<1$) and assign $\varepsilon = s\tau/8$. Then $\tau+\varepsilon \le \delta_*/2 + \delta_*/16 < \delta_*$ and $\varepsilon \le s$. Hence, by (5.37), $\theta^1_A(x,sr) < \tau$. Thus we have proved that if $\theta^d_A(x,r) < \varepsilon$ and $\theta^1_A(x,r) < \tau$, then on a smaller scale $\theta^1_A(x,sr) < \tau$, as well. Note that here $s = s(n,d,\delta) < 1$.

To finish the lemma, we now suppose that $A \subset \mathbb{R}^n$, $x \in A$ and $r>0$ satisfy (5.28) and suppose that

$$\theta^1_A(x,r) < \eta := s\tau/6. \tag{5.38}$$

Then, by Lemma 2.10,

$$\theta^1_A(x,tr) < 6\eta/t < 6\eta/s = \tau \quad \text{for all } s < t \le 1. \tag{5.39}$$

Since $\theta^d_A(x,tr) < \varepsilon$ (by (5.28)) and $\theta^1_A(x,tr) < \tau$ for all $s < t \le 1$, the argument above implies $\theta^1_A(x,str) < \tau$ for all $s<t\le 1$, or equivalently,

$$\theta^1_A(x,tr) < \tau \quad \text{for all } s^2 < t \le 1. \tag{5.40}$$

After a simple inductive argument, we conclude that $\theta^1_A(x,tr) < \tau$ for all $0 < t \le 1$. Therefore, since $\tau < \delta/2$, $\sup_{0<r'<r} \theta^1_A(x,r') < \delta$, as desired. $\square$

Theorem 5.23. *Let Ω be as in Corollary 5.17. Then Γ_1 is a relatively open subset of $\partial\Omega$.*

Proof. Let Ω be as in Corollary 5.17 and let $Q_0 \in \Gamma_1$. Let $\delta_{n,2}, \dots, \delta_{n,d}$ be the constants from Theorem 3.2. And let $\varepsilon = \varepsilon(n, d, \delta)$ and $\eta = \eta(n, d, \delta)$ be constants from Lemma 5.22 which correspond to $\delta = \min(\delta_{n,2}, \dots, \delta_{n,d})/2$. By Lemma 5.21, there exists $r_0 \in (0, 1)$ such that $\theta^d_{\partial\Omega}(Q, r) \leq \varepsilon$ for every $Q \in \partial\Omega \cap B(Q_0, 1)$ and for all $r \in (0, r_0)$. Since $Q_0 \in \Gamma_1$, $\lim_{r\downarrow 0} \theta^1_{\partial\Omega}(Q_0, r) = 0$. Hence we can find $r_1 \in (0, r_0/2)$ such that $\theta^1_{\partial\Omega}(Q_0, 2r_1) < \eta/12$. Thus $\theta^1_{\partial\Omega}(Q, r_1) < \eta$ for every $Q \in \partial\Omega \cap B(Q_0, r_1)$ by Lemma 2.10. Using Lemma 5.22, we conclude $\theta^1_{\partial\Omega}(Q, r') < \delta$ for all $Q \in \partial\Omega \cap B(Q_0, r_1)$ and for all $r' < r_1$.

Fix $Q \in \partial\Omega \cap B(Q_0, r_1)$. Let $k = k(Q)$ be the unique integer such that $Q \in \Gamma_k$. Since $Q \in \Gamma_k$ and $r_1/j \to 0$ as $j \to \infty$, together Lemma 2.18 and Theorem 5.19 yield a homogeneous harmonic polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree k such that

$$\frac{\partial\Omega - Q}{r_1/j} \cap B_1 \to \Sigma_p \cap B_1 \tag{5.41}$$

along some subsequence $j \to \infty$. Moreover, $\theta^1_{\Sigma_p}(0, 1) \leq \liminf \theta^1_{\partial\Omega}(Q, r_1/j) \leq \delta_{n,k}/2$. By Theorem 3.2, $Dp(0) \neq 0$. But since p is homogeneous, $Dp(0) \neq 0$ if and only if $k = \deg p = 1$. Therefore, $Q \in \Gamma_1$ for every $Q \in \partial\Omega \cap B(Q_0, r_1)$ and Γ_1 is open. $\square$

Corollary 5.24. *Let Ω be as in Corollary 5.17. Then Γ_1 is locally Reifenberg vanishing.*

Proof. Let $\Omega \subset \mathbb{R}^n$ be as in Corollary 5.17, let $\delta > 0$ and let $K \subset \Gamma_1$ compact be given. Let $\varepsilon = \varepsilon(n, d, \delta) > 0$ and $\eta = \eta(n, d, \delta) > 0$ be the constants from Lemma 5.22. Since Γ_1 is open and K is compact, we can find $r_0 > 0$ such that $\partial\Omega \cap B(Q, r) = \Gamma_1 \cap B(Q, r)$ for all $Q \in K$ and for all $r < r_0$. By Lemma 5.21, there exists $r_1 \in (0, r_0)$ such that $\theta^d_{\partial\Omega}(Q, r) < \varepsilon$ for all $Q \in K$ and $r \in (0, r_0)$. Since $K \subset \Gamma_1$, for each $Q \in K$ there exists $r_Q \in (0, r_1)$ with $\theta^1_{\partial\Omega}(Q, 2r_Q) < \eta/12$. Hence, by Lemma 2.10, $\theta^1_{\partial\Omega}(Q', r_Q) < \eta$ for all $Q' \in \partial\Omega \cap B(Q, r_Q)$, for all $Q \in K$. Thus, by Lemma 5.22,

$$\theta^1_{\partial\Omega}(Q', r') < \delta \quad \text{for all } Q' \in \partial\Omega \cap B(Q, r_Q), \text{ for all } r' \in (0, r_Q), \text{ for all } Q \in K. \tag{5.42}$$

But K is compact, so K admits a finite cover of the form $\{B(Q_i, r_{Q_i}) : Q_1, \ldots, Q_m \in K\}$. Letting $r_* = \min\{r_{Q_1}, \ldots, r_{Q_m}\}$, we conclude

$$\theta^1_{\Gamma_1}(Q, r') = \theta^1_{\partial\Omega}(Q, r') < \delta \quad \text{for all } Q \in K, \text{ for all } r' \in (0, r_*). \tag{5.43}$$

Thus, since $K \subset \Gamma_1$ was an arbitrary compact set, Γ_1 is locally δ-Reifenberg flat. Therefore, since $\delta > 0$ was arbitrary, Γ_1 is locally Reifenberg flat with vanishing constant. □

Corollary 5.25. *Let Ω be as in Corollary 5.17. Then Ψ–$\mathrm{Tan}(\omega, Q) \subset \mathcal{F}_1$ for all $Q \in \Gamma_1$.*

Proof. Let $Q \in \Gamma_1$ and let $\nu \in \Psi$–$\mathrm{Tan}(\omega, Q)$. By Theorem 5.10, the support of ν is a limit of $r_i^{-1}(\partial\Omega - Q_i)$ in the Hausdorff distance for some $Q_i \to Q$ and $r_i \to 0$. Hence $\operatorname{spt}\nu$ is a hyperplane, since $\partial\Omega$ is Reifenberg vanishing near Q by Corollary 5.24. Thus, we conclude $\nu \in \mathcal{F}_1$ for every $Q \in \Gamma_1$. □

The decomposition of the boundary in Corollary 5.17 has an extra interpretation from the geometric measure theory viewpoint. Unfortunately the proof of Theorem 2.29 does not provide a certificate to check at which points in the support of a measure the translations of tangent measures are tangent measures. But the corollary identifies the points in the support of harmonic measure where this behavior occurs.

Proposition 5.26 ([3] Proposition 6.8)**.** *Let Ω be as in Corollary 5.17. Then the cone $\mathrm{Tan}(\omega^\pm, Q)$ is translation invariant if and only if $Q \in \Gamma_1$.*

Proof. If μ is a flat measure, then $T_{x,1}[\mu] = \mu$ for every $x \in \operatorname{spt}\mu$. Hence $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_1$ is translation invariant for every $Q \in \Gamma_1$.

Conversely, assume $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ is translation invariant and let $\nu \in \mathrm{Tan}(\omega^\pm, Q)$. Then $\operatorname{spt}\nu = h^{-1}(0)$ for some harmonic polynomial h. By [12] the zero set of a harmonic polynomial is smooth away from a rectifiable subset of dimension at most $n-2$. Hence $\operatorname{spt}\nu$ is smooth at some $x \in \operatorname{spt}\nu$. Because $\psi := T_{x,1}[\nu] \in \mathrm{Tan}(\omega^\pm, Q)$ and $\operatorname{spt}\psi = \operatorname{spt}\nu - x$, we conclude there exists $\psi \in \mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ such that $\operatorname{spt}\psi$ is smooth at the origin. But the zero set of a non-zero homogeneous polynomial of degree k (i.e. the support of ψ) is smooth at the origin if and only if $k = 1$. Therefore, $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_1$ and $Q \in \Gamma_1$. □

Corollary 5.27 ([3] Corollary 6.9)**.** *Let* Ω *be as in Corollary 5.17. Then* $\omega^{\pm}(\partial\Omega \setminus \Gamma_1) = 0$.

Proof. By Theorem 2.29, the cone $\mathrm{Tan}(\omega^{\pm}, Q)$ of tangent measures of $\omega^{\pm}$ at Q is translation invariant for $\omega^{\pm}$-a.e. $Q \in \partial\Omega$. Since this property fails at all $Q \in \partial\Omega \setminus \Gamma_1$, the set must have zero harmonic measure. □

We have collected all the ingredients to record:

Proof of Structure Theorem for FBP 2. Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain such that $\omega^+ \ll \omega^- \ll \omega^+$ and $\log d\omega^-/d\omega^+ \in \mathrm{VMO}(d\omega^+)$. By Corollary 5.17 we can write $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d$ where $\mathrm{Tan}(\omega^{\pm}, Q) \subset \mathcal{F}_k$ for all $Q \in \Gamma_k$ (and d only depends on n and the NTA constants of Ω). By Theorem 5.19 every blow-up of $\partial\Omega$ centered at $Q \in \Gamma_k$ is the zero set of a homogeneous harmonic polynomial $h : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{R}^n \setminus \Sigma_h$ has two connected components. By Theorem 5.23, Γ_1 is a relatively open subset of the boundary; and by Corollary 5.24, Γ_1 is locally Reifenberg flat with vanishing constant. Finally, by Corollary 5.27, $\omega^{\pm}(\partial\Omega \setminus \Gamma_1) = \omega^{\pm}(\Gamma_2 \cup \cdots \cup \Gamma_d) = 0$. □

To conclude we give the example promised in Remark 2.36. Our goal is to show that the cone of pseudotangent measures is *not* connected in the topology of weak convergence of Radon measures, in the sense that Theorem 2.35 has no analogue for pseudotangents.

Example 5.28. Let $\Omega = \{x_1^2 + x_2^2 - x_3^2 - x_4^2 > 0\} \subset \mathbb{R}^4$. Then $\Omega \subset \mathbb{R}^4$ is a 2-sided NTA domain, $\omega^+ \ll \omega^- \ll \omega^+$ and $f = d\omega^-/d\omega^+$ satisfies $\log f \equiv 0$. Moreover, $\partial\Omega = \Gamma_1 \cup \Gamma_2$ where $\mathrm{Tan}(\omega^{\pm}, Q) \subset \mathcal{F}_k$ for all $Q \in \Gamma_k$, $k = 1, 2$. In fact, since $\partial\Omega$ is smooth away from the origin, $\Gamma_1 = \partial\Omega \setminus \{0\}$ and $\Gamma_2 = \{0\}$. Because every tangent measure is a pseudotangent measure,

$$\Psi\text{–}\mathrm{Tan}(\omega, 0) \cap \mathcal{F}_2 \supset \mathrm{Tan}(\omega, 0) \cap \mathcal{F}_2 \neq \emptyset. \tag{5.44}$$

We also claim that in this circumstance

$$\Psi\text{–}\mathrm{Tan}(\omega, 0) \cap \mathcal{F}_1 \neq \emptyset. \tag{5.45}$$

To see this, choose any sequence $Q_i \in \partial\Omega \setminus \{0\}$ such that $Q_i \to 0$. For each $i \geq 1$, we can find a flat measure $\mu_i \in \mathcal{F}_1$ (i.e. a tangent measure of ω at $Q_i \in \Gamma_1$) such that $F_1(\mu_i) = 1$, a scale $0 < r_i < 1/i$ and a constant $c_i > 0$ such that

$$F_i(c_i T_{Q_i,r_i}[\omega], \mu_i) \leq 1/i. \tag{5.46}$$

Since $\mathcal{F}_1$ has a compact basis (Lemma 4.11), there exists a flat measure $\mu \in \mathcal{F}_1$ and a subsequence of $(\mu_i)_{i=1}^{\infty}$ (which we relabel) such that $\mu_i \rightharpoonup \mu$. If $s > 0$, then for all $i \geq \lceil s \rceil$,

$$F_s(c_i T_{Q_i,r_i}[\omega], \mu) \leq F_s(c_i T_{Q_i,r_i}[\omega], \mu_i) + F_s(\mu_i, \mu) \leq \frac{1}{i} + F_s(\mu_i, \mu). \tag{5.47}$$

Letting $i \to \infty$ in (5.47), we conclude $\lim_{i\to\infty} F_s(c_i T_{Q_i,r_i}[\omega], \mu) = 0$ for all $s > 0$. Hence, $\mu \in \Psi\text{–}\mathrm{Tan}(\omega, 0) \cap \mathcal{F}_1$. This establishes (5.45).

On the other hand, the cones $\mathcal{M} = \Psi\text{–}\mathrm{Tan}(\omega, 0) \cup \mathcal{F}_1$ and $\mathcal{F} = \mathcal{F}_1$ satisfy the hypotheses of Theorem 2.35. In particular, $\mathcal{F} \subset \mathcal{M}$, the cones $\mathcal{F}$ and $\mathcal{M}$ both have compact bases (since ω is locally doubling) and there exists $\varepsilon > 0$ such that $\mu \in \mathcal{M}$ and $d_r(\mu, \mathcal{F}) < \varepsilon$ for all $r \geq r_0$ implies $\mu \in \mathcal{F}$ (by Corollary 4.10). If the conclusion of Theorem 2.35 held for pseudotangent measures, then we could conclude that $\Psi\text{–}\mathrm{Tan}(\omega, 0) \subset \mathcal{F}_1$, by (5.45). But this would contradict (5.44). Therefore, Theorem 2.35 has no analogue for the cone of pseudotangent measures.

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Appendix A

LOWER BOUND FOR THE DIMENSION OF HARMONIC MEASURE

We present a simple proof that harmonic measure of any bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ has lower Hausdorff dimension at least $n-2$. The proof is geometric and avoids the notion of capacity. Although Theorem A (and in fact a stronger theorem) is certainly known to experts, the author is unaware of an easy reference to this result in the literature. We hope that by recording this proof, the lower bound may become more widely known. Below ω_Ω^X denotes the harmonic measure of a domain $\Omega \subset \mathbb{R}^n$ with pole at $X \in \Omega$.

Theorem A. *If $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain and $X \in \Omega$, then $\omega_\Omega^X \ll \mathcal{H}^{n-2}$.*

For any $Q \in \mathbb{R}^n$ and $0 < r < R$, let $S(Q,r) = \{X \in \mathbb{R}^n : |X-Q| = r\}$ denote the inner shell of the annulus $A(Q,r,R) = \{X \in \mathbb{R}^n : r < |X-Q| < R\}$.

Lemma A.1. *If $n \geq 3$ and $X \in A(Q,r,R)$, then*

$$\omega^X_{A(Q,r,R)}(S(Q,r)) = \frac{|X-Q|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}.$$

Proof. The right hand side is harmonic on $\mathbb{R}^n \setminus \{Q\}$, identically one on the inner shell $S(Q,r)$ and identically zero on the outer shell $S(Q,R) = \partial A(Q,r,R) \setminus S(Q,r)$. □

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain. If $X \in \Omega \setminus B(Q,r)$, then*

$$\omega_\Omega^X(\partial\Omega \cap B(Q,r)) \leq \frac{r^{n-2}}{\operatorname{dist}(X,\partial\Omega)^{n-2}}.$$

Proof. Let $Q \in \partial\Omega$ and fix $R \gg \operatorname{diam}\Omega$. By the maximum principle and Lemma A.1,

$$\omega_\Omega^X(\partial\Omega \cap B(Q,r)) \leq \omega^X_{A(Q,r,R)}(S(Q,r)) = \frac{|X-Q|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}}$$

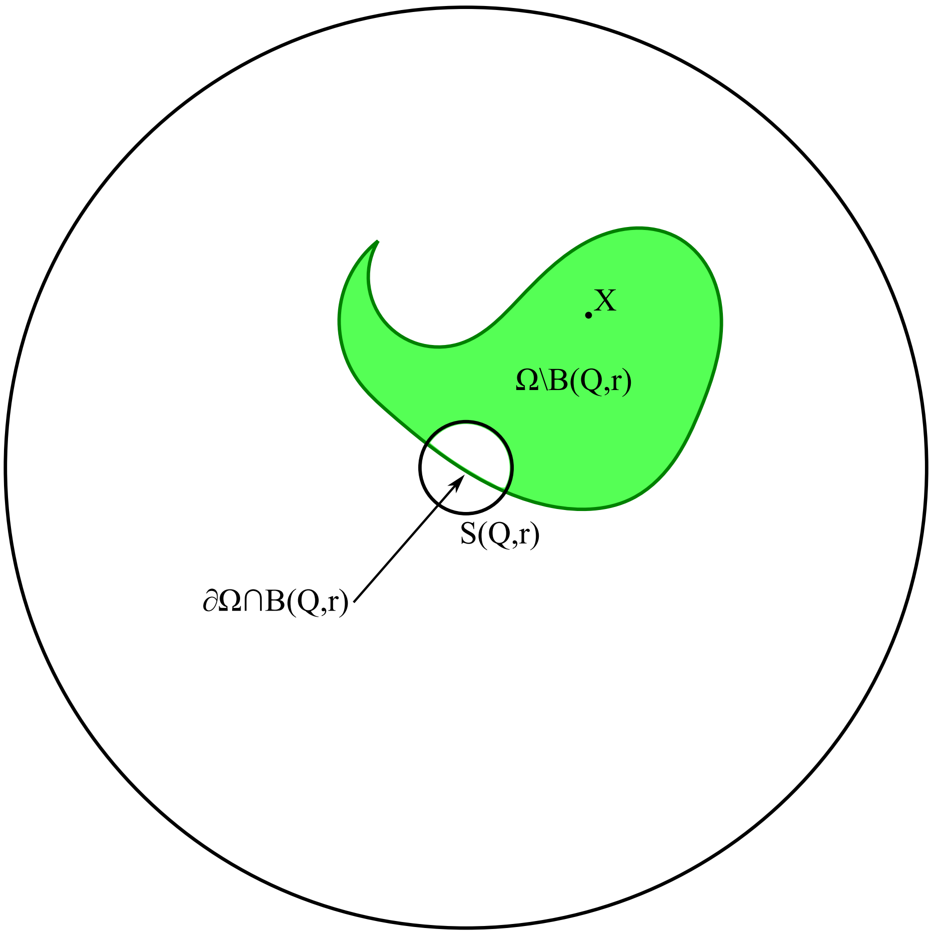

Figure A.1: Annulus Centered on the Boundary of a Domain

whenever $X \in \Omega \setminus B(Q,r)$ (see Figure A.1). Letting $R \to \infty$ yields

$$\omega_\Omega^X(\partial\Omega \cap B(Q,r)) \leq \frac{|X-Q|^{2-n}}{r^{2-n}} \leq \frac{\operatorname{dist}(X,\partial\Omega)^{2-n}}{r^{2-n}} = \frac{r^{n-2}}{\operatorname{dist}(X,\partial\Omega)^{n-2}}$$

as desired. $\square$

The proof of Theorem A is immediate from Lemma A.2 and (2.1), (2.3) and (2.4). Thus, the lower Hausdorff dimension $\underline{\dim}_H\, \omega_\Omega^X \geq n-2$, because any set $E \subset \partial\Omega$ with $\omega_\Omega^X(E) > 0$ has $\mathcal{H}^{n-2}(E) > 0$, and hence $\dim_H E \geq n-2$, by Theorem A.