Free Boundary Regularity for Harmonic Measure from Two Sides

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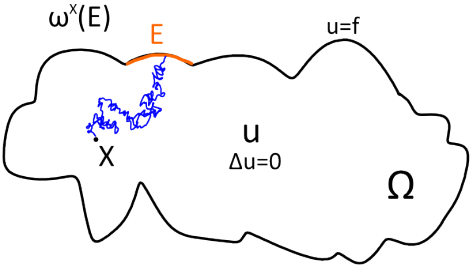
Dirichlet Problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.



Dirichlet Problem

$$(D)\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \\ u \in C(\partial\Omega) \cap C^2(\Omega) \\ f \in C_c(\partial\Omega) \end{cases}$$

$\exists!$ probability measure ω^X on $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

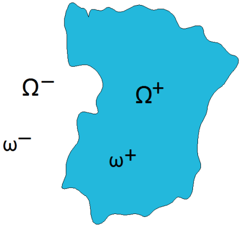
$$u(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q) \qquad \text{solves (D)}$$

Harmonic Measure on 2-sided Domains

Let $\Omega \subset \mathbb{R}^n$ be 2-sided domain

ω^+ harmonic measure of interior $\Omega^+ = \Omega$

ω^- harmonic measure of exterior $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$

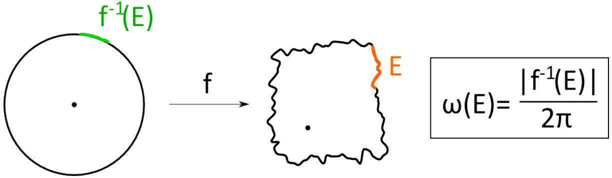


Question: What is the relationship between ω^+ and ω^-?

- When are ω^+ and ω^- mutually absolutely continuous?
  (e.g. smooth domains, Lipschitz domains, chord arc domains)
- What conditions does $\omega^+ \ll \omega^- \ll \omega^+$ force on $\partial\Omega$?
- When is $\omega^+ \perp \omega^-$?
- What are the possible Hausdorff dimensions of ω^+ and ω^-?

The Case $n = 2$: Jordan Domains

Let $\Omega^+ \subset \mathbb{R}^2$ and $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$ be simply connected domains.
($\omega^\pm$ = harmonic measure on $\Omega^\pm$)



- (McMillan 1969, Makarov 1985, Pommerenke 1986)
  The following are equivalent:
  - $\omega^+ \ll \omega^- \ll \omega^+$
  - $\partial\Omega = G \cup N$: $\omega^\pm(N) = 0$ and $\omega^\pm \llcorner G \ll \mathcal{H}^1 \llcorner G \ll \omega^\pm \llcorner G$
    [$\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure (i.e. length)]
- (Bishop-Carleson-Garnett-Jones 1989) $\omega^+ \perp \omega^-$ if and only if $\mathcal{H}^1(\{Q \in \partial\Omega^\pm\colon \partial\Omega^\pm \text{ has a unique tangent line}\}) = 0$

Dimension of Harmonic Measure

Definition: The (upper) Hausdorff dimension of harmonic measure

$$\dim \omega = \inf\{\dim E : E \subset \partial\Omega \text{ and } \omega(\partial\Omega \setminus E) = 0\}$$

is the smallest dimension of a set with full harmonic measure

- (Makarov 1985) $\dim \omega = 1$ if $\Omega \subset \mathbb{R}^2$ is simply connected
- (Wolff 1995) – There are NTA domains in $\mathbb{R}^3$ with $\dim \omega > 2$
  – There are NTA domains in $\mathbb{R}^3$ with $\dim \omega < 2$
- (Lewis-Verchota-Vogel 2005) Reexamined Wolff's construction: For all $n \geq 3$ there are 2-sided NTA domains in $\mathbb{R}^n$ such that
  - $\dim \omega^+ > n - 1$, $\dim \omega^- > n - 1$
  - $\dim \omega^+ > n - 1$, $\dim \omega^- < n - 1$
  - $\dim \omega^+ < n - 1$, $\dim \omega^- < n - 1$
- (Kenig-Preiss-Toro 2009) If Ω is 2-sided NTA and $\omega^+ \ll \omega^- \ll \omega^+$, then $\dim \omega^+ = \dim \omega^- = n - 1$

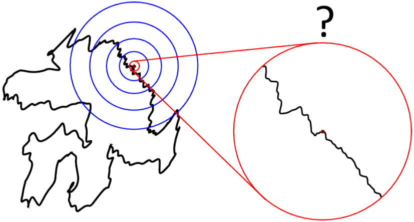
2-sided Free Boundary Problem

Non-technical Formulation: If ω^+ and ω^- are "comparable", what does the boundary look like?

Technical Formulation: Suppose Ω^+ and Ω^- are NTA domains, $\omega^+ \ll \omega \ll \omega^-$ and $\log f \in C(\partial\Omega)$ where

$$f(Q) = \frac{d\omega^-}{d\omega^+}(Q) = \lim_{r \downarrow 0} \frac{\omega^-(B(Q,r))}{\omega^+(B(Q,r))}.$$

Describe blow-ups of the boundary in the Hausdorff distance.



$Q \in \partial\Omega, \; \lim_{i\to\infty} r_i \to 0$

$$\lim_{i\to\infty} \frac{\partial\Omega - Q}{r_i} = \boxed{?}$$

Blow-ups of the Boundary are Homogeneous

Structure Theorem (B) Assume $\Omega \subset \mathbb{R}^n$ is 2-sided NTA, $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega)$.

There exists $d \geq 1$ (depending on the NTA constants) such that

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d.$$

1. Every blow-up of $\partial\Omega$ about a point $Q \in \Gamma_k$ is the zero set $h^{-1}(0)$ of a homogeneous harmonic polynomial h of degree k which separates $\mathbb{R}^n$ into two components.
2. The "flat points" Γ_1 is an open subset of $\partial\Omega$.
3. The "singularities" $\Gamma_2 \cup \cdots \cup \Gamma_d$ have harmonic measure zero.

Flat Points $Q \in \Gamma_1$

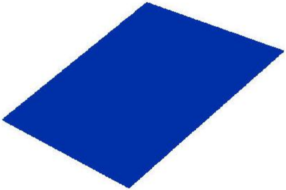


Figure: $h^{-1}(0)$ when h homogeneous harmonic polynomial of degree 1, i.e. a plane through the origin

Remarks

- $\Gamma_1 \subset \partial\Omega$ is open and $\omega^{\pm}(\partial\Omega \setminus \Gamma_1) = 0$.
- At $Q \in \Gamma_1$ one can see different planes as $\lim_{i\to\infty} \frac{\partial\Omega - Q}{r_i}$ along different sequences of scales $r_i \downarrow 0$.
- In dimension $n = 2$, $\partial\Omega = \Gamma_1$ and $\partial\Omega$ is Reifenberg flat with vanishing constant.

Singularities $Q \in \Gamma_2 \cup \cdots \cup \Gamma_d$

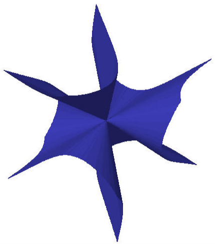


Figure: $h^{-1}(0)$ where $h(x, y, z) = x^2(y - z) + y^2(x - z) + z^2(x - y) - xyz$ is an example blow-up of $\partial\Omega$ about $Q \in \Gamma_3$, $\Omega \subset \mathbb{R}^3$.

Remarks

- $\Gamma_2 \cup \cdots \cup \Gamma_d \subset \partial\Omega$ is closed and $\omega^{\pm}(\Gamma_2 \cup \cdots \cup \Gamma_d) = 0$.
- Examples shows that $\dim(\Gamma_2 \cup \cdots \cup \Gamma_d) = n - 3$ is possible. Upper bound is unknown.
- In dimension $n = 3$, $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \Gamma_5 \cup \cdots \cup \Gamma_{2k+1}$.

(Lewy 1977)

Ingredients in the Proof

Initial Observations: (Kenig-Toro 2006)

- On NTA domain: Blow-ups of the Boundary $\longleftrightarrow$
  Tangent Measures of Harmonic Measure
- Under the hypothesis $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega)$, tangent measures of $\omega^\pm$ are "polynomial harmonic measures".

GMT Tools: (Preiss 1987, Kenig-Preiss-Toro 2009)

- Under certain conditions, the cone of tangent measures $\mathrm{Tan}(\mu, x)$ of a measure μ at $x \in \operatorname{spt} x$ is connected.

Ingredients in the Proof

If ω_h is a polynomial harmonic measure associated to a **homogeneous** harmonic polynomial of degree d, then

$$\frac{\omega_h(B(0,2r))}{\omega_h(B(0,r))} = 2^{n+d-2} \quad \text{for all } r > 0.$$

Key Estimate: (B)

- If ω_h is a polynomial harmonic measure associated to **any** harmonic polynomial of degree d, then for all $\tau > 1$

$$\frac{\omega_h(B(0,\tau r))}{\omega_h(B(0,r))} \sim \tau^{n+d-2} \quad \text{as } r \to \infty \qquad (\star)$$

where the implied constant in $(\star)$ only depend on n and d.

Extensions of Structure Theorem

Structure Theorem

$$\omega^+ \ll \omega^- \ll \omega^+, \log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega) \Rightarrow \boxed{\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d}$$

Remarks

- The theorem only requires $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^-)$, the space of functions of vanishing mean oscillation.

- If $\Omega \subset \mathbb{R}^n$ is 2-sided NTA and $\omega^+ \ll \omega^- \ll \omega^+$, then $\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d \cup N$ where Γ_k is as above and $\omega^\pm(N) = 0$.

Open Questions

Structure Theorem

$$\omega^+ \ll \omega^- \ll \omega^+, \log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega) \Rightarrow \boxed{\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d}$$

Open Questions

1. Can we write $\Gamma_1 = G \cup N$ where
   - G is $(n-1)$-rectifiable and $\omega^\pm \ll \mathcal{H}^{n-1} \ll \omega^\pm$ on G
   - $\omega^\pm(N) = 0$

   [The answer is yes if $\omega^+ \ll \omega^- \ll \omega^+$ **and** $\mathcal{H}^{n-1}(\partial\Omega) < \infty$]

2. Is $\dim \Gamma_k < n-1$ for $k \geq 2$?

3. Can we relax the NTA assumptions? Is there a version of the theorem on semi-uniform John domains?