Multiscale Analysis of 1-Rectifiable Measures

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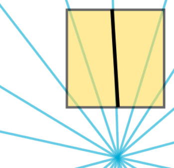

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General Definition

Let μ be a Borel measure on $\mathbb{R}^n$ and let $1 \le m \le n-1$.
We say that μ is *m-rectifiable* if there exist countably many

- Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$

such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_i f_i([0,1]^m)\right) = 0.$$

(Federer's terminology: $\mathbb{R}^n$ is countably (μ, m)-rectifiable.)

Examples

- rectifiable curves/surfaces: $\mathcal{H}^m \llcorner f([0,1]^m)$,
- (countably) rectifiable sets: $\sum_i \mathcal{H}^m \llcorner E_i, \quad E_i \subset f_i([0,1]^m)$
- Dirac mass δ_x at $x \in \mathbb{R}^n$

Surprising Example

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ $(n \geq 2)$ with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

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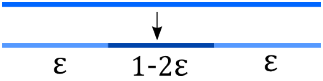

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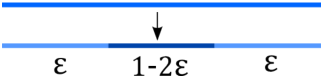

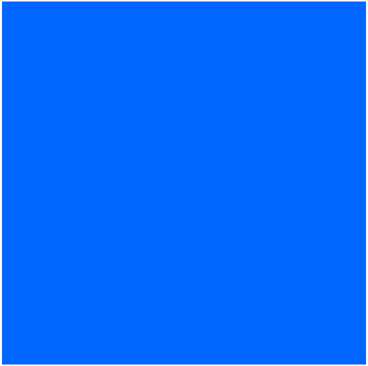

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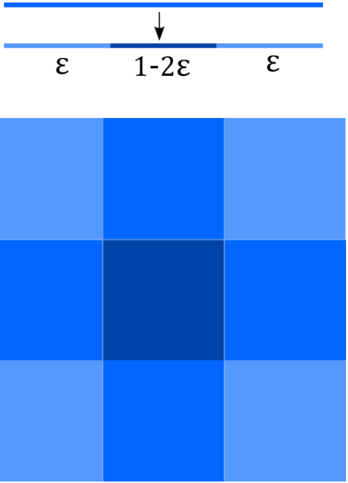

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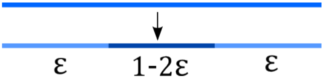

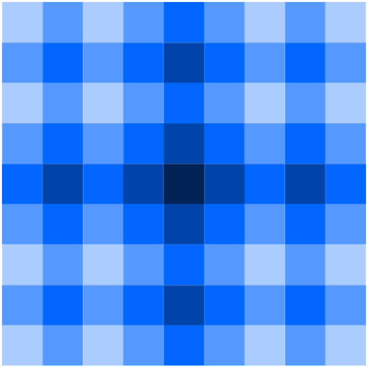

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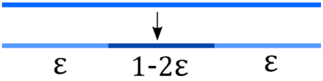

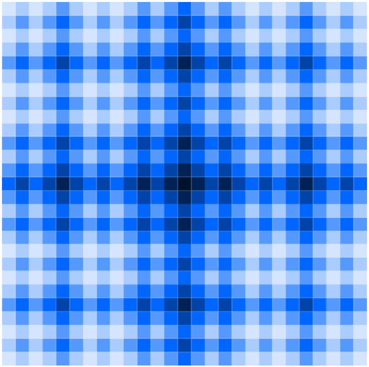

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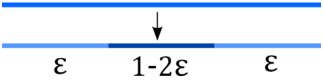

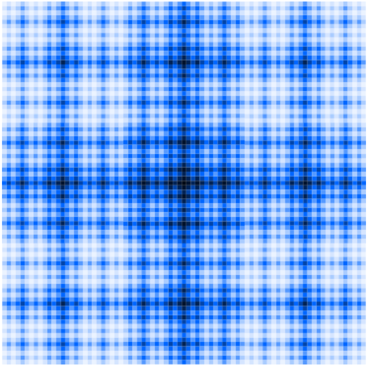

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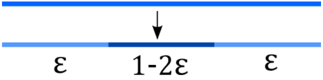
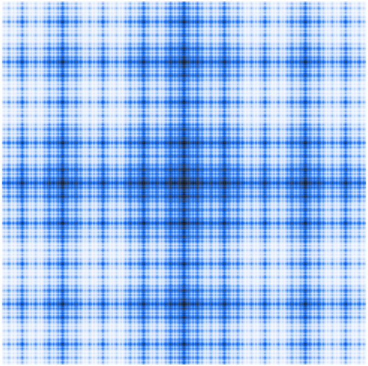




Grades of Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\cup\!\!\!\!\not\,$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

$$\cup\!\!\!\!\not\,$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E\ \}$$

Absolutely Continuous Rectifiable Measures

The lower and upper (Hausdorff) m-density of a measure μ at x:

$$\underline{D}^m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m} \quad \overline{D}^m(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m}.$$

Write $D^m(\mu, x)$, the m-density of μ at x, if $\underline{D}^m(\mu, x) = \overline{D}^m(\mu, x)$.

Theorem (Mattila 1975)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m \llcorner E$ is locally finite. Then μ is m-rectifiable if and only if $D^m(\mu, x) = 1$ μ-a.e.

Theorem (Preiss 1987)

Suppose that μ is a locally finite Borel measure on $\mathbb{R}^n$. Then μ is m-rectifiable and $\mu \ll \mathcal{H}^m$ if and only if $0 < D^m(\mu, x) < \infty$ μ-a.e.

There are additional characterizations (tangent measures, etc.)

General Rectifiable Measures

Problem

For all $1 \leq m \leq n-1$, find necessary and sufficient conditions for a locally finite Borel measure μ on $\mathbb{R}^n$ to be m-rectifiable.

- Do not assume $\mu \ll \mathcal{H}^m$.

Theorem (B-Schul)

Necessary condition for the case $m = 1$ and $n \geq 2$:

If μ is 1-rectifiable, then at μ-almost every $x \in \mathbb{R}^n$,

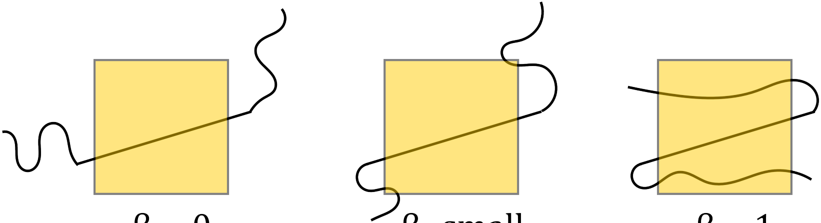
- *$\mu \llcorner B(x,r)$ concentrates mass around a line $\ell_{x,r}$ as $r \to 0$; or*
- *the density $\mu(B(x,r))/r \to \infty$ sufficiently fast as $r \to 0$.*

L^2 Beta Numbers

Let μ be a locally finite Borel measure on $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ a cube. Define the L^2 beta number $\beta_2^2(\mu, Q) \in [0,1]$ by

$$\beta_2^2(\mu, Q) = \inf_{\ell} \int_Q \left(\frac{\operatorname{dist}(x, \ell)}{\operatorname{diam} Q} \right)^2 \frac{d\mu(x)}{\mu(Q)}$$

where the infimum runs over all lines ℓ in $\mathbb{R}^n$.



$\beta_2 = 0$ β_2 small $\beta_2 \sim 1$

L^2 Jones Functions

A collection $\{w(\mu, Q)\}$ of weights $\rightsquigarrow$ weighted L^2 Jones function:

$$J_2^w(\mu, r, x) = \sum_{\substack{\text{side } Q \leq r \\ Q \text{ dyadic}}} \beta_2^2(\mu, 3Q) w(\mu, Q) \chi_Q(x).$$

Two Special Cases

$w(\mu, Q) \equiv 1 \rightsquigarrow$ **ordinary L^2 Jones function**

$$J_2(\mu, r, x) = \sum_{\substack{\text{side } Q \leq r \\ Q \text{ dyadic}}} \beta_2^2(\mu, 3Q) \chi_Q(x).$$

$w(\mu, Q) \equiv \left(\frac{\mu(Q)}{\text{diam } Q}\right)^{-1} \rightsquigarrow$ **density-normalized L^2 Jones function**

$$\tilde{J}_2(\mu, r, x) = \sum_{\substack{\text{side } Q \leq r \\ Q \text{ dyadic}}} \beta_2^2(\mu, 3Q) \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x).$$

Ordinary Jones Function and Rectifiable Sets

A Borel measure μ on $\mathbb{R}^n$ is m-Ahlfors regular if $\mu(B(x,r)) \sim r^m$ for all x in the support of μ and for all $0 < r < r_0(\mu)$.

Theorem (David-Semmes 1991)

Suppose $E \subset \mathbb{R}^n$ is closed and $\mu = \mathcal{H}^m \llcorner E$ is m-AR. Then μ is uniformly m-rectifiable if and only if

$$\int_{B(x_0,r)} J_2(\mu, r, x)\,d\mu(x) \lesssim r^m \quad \textit{for all } x_0 \in E,\ 0 < r < \operatorname{diam} E.$$

Theorem (Pajot 1997)

Suppose $K \subset \mathbb{R}^n$ is compact and $\mu = \mathcal{H}^m \llcorner K$.

- *Suppose μ is m-AR. Then μ is m-rectifiable if and only if $J_2(\mu, x) < \infty$ μ-a.e.*
- *Suppose $\mathcal{H}^m(K) < \infty$. Then μ is m-rectifiable if both $\underline{D}^m(\mu, x) > 0$ and $J_2(\mu, x) < \infty$ μ-a.e.*

Necessary Conditions for 1-Rectifiable Measures

Theorem (B-Schul)

Let μ be a locally finite Borel measure on $\mathbb{R}^n$.

- *If μ is 1-rectifiable, then*

$$\widetilde{J}_2(\mu, x) = \sum_{\substack{\text{side } Q \leq 1 \\ Q \text{ dyadic}}} \beta_2^2(\mu, 3Q) \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

- *If μ is 1-rectifiable and $\mu \ll \mathcal{H}^1$, then*

$$J_2(\mu, x) = \sum_{\substack{\text{side } Q \leq 1 \\ Q \text{ dyadic}}} \beta_2^2(\mu, 3Q) \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

Corollary (B-Schul + Pajot 1997)

Suppose $K \subset \mathbb{R}^n$ is compact and $\mathcal{H}^1(K) < \infty$. Then $\mu = \mathcal{H}^1 \llcorner K$ 1-rectifiable if and only if $\underline{D}^1(\mu, x) > 0$ and $J_2(\mu, x) < \infty$ μ-a.e.

Heart of the Theorem

Proposition (B-Schul)

Suppose $\nu(\mathbb{R}^n) < \infty$, $\Gamma \subset \mathbb{R}^n$ *is a rectifiable curve,* $E \subset \Gamma$ *is Borel and* $\nu(E \cap B(x,r)) \geq cr$ *for all* $x \in E$ *and* $0 < r \leq r_0$*. Then*

$$\int_E \widetilde{J}_2(\nu, r_0, x)\, d\nu(x) \lesssim_{n,c} \mathcal{H}^1(\Gamma) + \nu(\mathbb{R}^n \setminus \Gamma).$$

- $\int_E \widetilde{J}_2(\nu, r_0, x)\, d\nu(x) = \sum_{\substack{\mathrm{side}\, Q \leq r_0 \\ Q \text{ dyadic}}} \beta_2^2(\nu, 3Q)\, \mathrm{diam}\, Q \frac{\nu(E\cap Q)}{\nu(Q)}$
- Dyadic cubes Q with $\nu(E \cap Q) > 0 \rightsquigarrow$ two classes: $\{\beta_2^2(\nu, 3Q) \lesssim \beta_\Gamma(3Q)\}$ and $\{\beta_\Gamma(3Q) \ll \beta_2^2(\nu, 3Q)\}$
- Sum over first class $\lesssim_n \mathcal{H}^1(\Gamma)$: Traveling Salesman Theorem for Rectifiable Curves (Jones 1990 in $\mathbb{R}^2$, Okikiolu 1992 in $\mathbb{R}^n$)
- Sum over second class $\lesssim_{n,c} \nu(\mathbb{R}^n \setminus \Gamma)$: New Estimate!

Future Directions

Problem

For all $1 \leq m \leq n-1$, find necessary and sufficient conditions for a locally finite Borel measure μ on $\mathbb{R}^n$ to be m-rectifiable.

- Do not assume $\mu \ll \mathcal{H}^m$.

Necessary Conditions

- If μ is m-rectifiable, then $\underline{D}^m(\mu, x) > 0$ μ-a.e.
- If μ is 1-rectifiable, then $\widetilde{J}_2(\mu, x) < \infty$ μ-a.e. (B-Schul)
- What happens for m-rectifiable measures, $m \geq 2$?

Sufficient Conditions

- If $\underline{D}^1(\mu, x) > 0$ and $\widetilde{J}_2(\mu, x) < \infty$ μ-a.e., is μ 1-rectifiable?
- Same ? is open for $\mu \ll \mathcal{H}^1$ (but settled for $\mu = \mathcal{H}^1 \llcorner K$).