Quasiconformal Planes and Bi-Lipschitz Parameterizations

Joint work with
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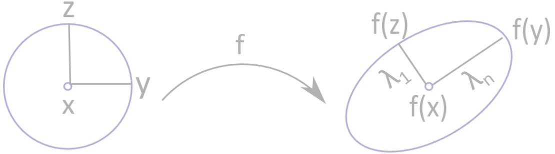
Two Measurements of Distortion of a Map

$f : \Omega \xrightarrow{\sim} \Omega'$ a homeomorphism of domains in $\mathbb{R}^n$, $f \in W^{1,n}_{loc}(\Omega)$



If $Df(x)$ exists and has singular values $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$, then

$$K_f(x) = \max\left(\frac{\lambda_n(x)}{\lambda_1(x)}\frac{\lambda_n(x)}{\lambda_2(x)}\cdots\frac{\lambda_n(x)}{\lambda_n(x)}, \frac{\lambda_1(x)}{\lambda_1(x)}\frac{\lambda_2(x)}{\lambda_1(x)}\cdots\frac{\lambda_n(x)}{\lambda_1(x)}\right)$$

The **maximal dilatation** (local distortion)

$$K_f(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} K_f(x) \in [1,\infty]$$

The **weak quasisymmetry constant** (global distortion)

$$H_f(\Omega) = \max\left\{\frac{|f(y)-f(x)|}{|f(z)-f(x)|} : x,y,z \in \Omega, \frac{|y-x|}{|z-x|} \leq 1\right\} \in [1,\infty]$$

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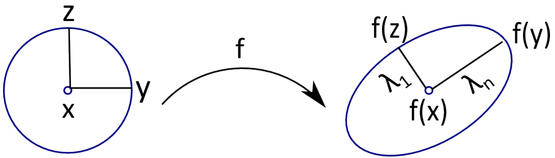

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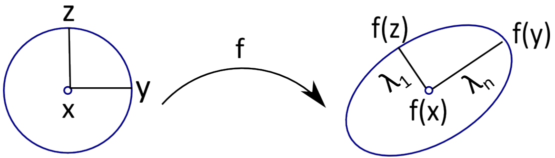

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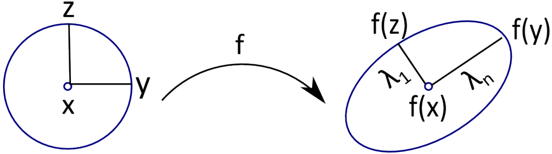

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Local Distortion versus Global Distortion

$f : \Omega \xrightarrow{\sim} \Omega'$ a homeomorphism of domains in $\mathbb{R}^n$, $f \in W^{1,n}_{loc}(\Omega)$

For all $n \geq 2$ and all domains $\Omega \subset \mathbb{R}^n$,

$$K_f(\Omega) \leq H_f(\Omega)^{n-1}$$

When $n \geq 2$ and $\Omega = \mathbb{R}^n$,

$$H_f(\mathbb{R}^n) - 1 \leq \Phi_n(K_f(\mathbb{R}^n) - 1), \quad \Phi_n : [0,\infty) \xrightarrow{\sim} [0,\infty)$$

- Precise formula for $\Phi_2(t)$ — Lehto, Virtanen, Väisälä 1959.
- $\Phi_n(0) = 0$ ($n \geq 3$) — Vuorinen 1989.

When $n \geq 2$, $\Omega = \mathbb{R}^n$ and K is near 1,

$$\Phi_2(K-1) \leq C_2(K-1) \quad (n=2)$$

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- Estimate ($n \geq 3$) by Seittenranta 1996 (cf. Prause 2007)
  It is not known if the logarithm term is necessary.

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Estimate for $H_f(B(z,s))$ when $K_f(B(z,Rs))$ is near 1

$f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ a homeomorphism of $\mathbb{R}^n$ and $f \in W^{1,n}_{loc}(\mathbb{R}^n)$

Theorem (B., Gill, Rohde, Toro 2012)

Given $n \geq 2$ and $1 < K \leq \min\{4/3, K'\}$, set

$$R = \left(\frac{c}{K-1}\right)^{c/(K-1)} > 1,$$

where $c > 1$ is a constant that only depends on n and K'. If $K_f(\mathbb{R}^n) \leq K'$ and $K_f(B(z,Rs)) \leq K$, then

$$H_f(B(z,s)) - 1 \leq C(K-1)\log\left(\frac{1}{K-1}\right),$$

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Quasiconformal Planes

A map $f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ is K-**quasiconformal** if

- f is a homeomorphism,
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A **quasiplane** $\Sigma = f(\mathbb{R}^m)$ is image of $\mathbb{R}^m$ under QC map of $\mathbb{R}^n$ $(1 \le m \le n-1)$. The **codimension** of Σ is $n - m$.

Examples:

General Question: What is the relationship between the distortion of f near $\mathbb{R}^m$ and the geometry of the quasiplane $\Sigma = f(\mathbb{R}^m)$?

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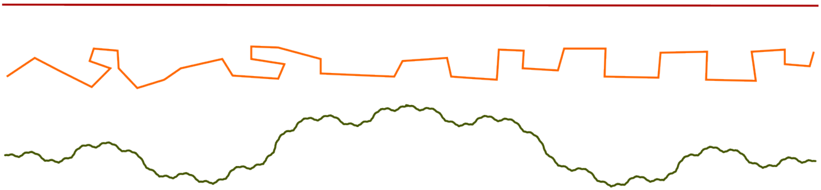

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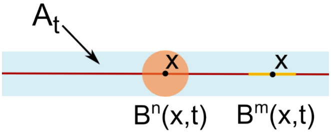
Asymptotically Conformal Quasiplanes

How is the geometry of $\Sigma = f(\mathbb{R}^m)$ controlled by the distortion of the map f near $\mathbb{R}^m$?



Let A_t be tubular neighborhood of $\mathbb{R}^m$ of size t.

$\Sigma = f(\mathbb{R}^m)$ is **asymptotically conformal** if $K_f(A_t) \to 1$ as $t \to 0$.

- If $K_f(\mathbb{R}^n) = 1$, then $f(\mathbb{R}^m)$ is an m-plane
- If $f(\mathbb{R}^m)$ is asymptotically conformal, then $\dim_H f(\mathbb{R}^m) = m$
- There exist asymptotical conformal $f(\mathbb{R}^m)$ such that $\mathcal{H}^m \llcorner f(\mathbb{R}^m)$ is locally infinite (e.g. "flat snowflakes")
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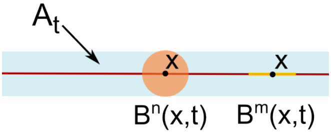

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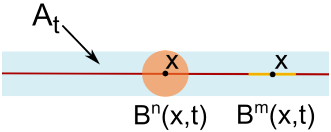

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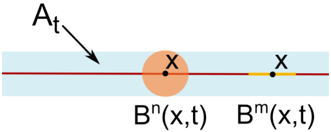

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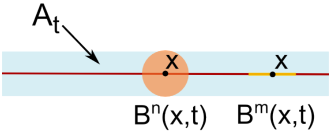

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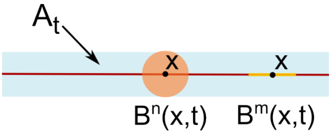
Rectifiability of $\Sigma = f(\mathbb{R}^m)$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiconformal and assume that

$$\int_0^{t_0} \Psi(K_f(A_t) - 1)\frac{dt}{t} < \infty.$$



Theorem (Carleson 1967, Anderson, Becker, Lesley 1988)

If $\Psi(t) = t^2$, $n = 2$, then $\mathcal{H}^1 \llcorner f(\mathbb{R}^1)$ is locally finite. Examples show that the conclusion may fail when $\Psi(t) = t^{2+\varepsilon}$.

Theorem (Mattila and Vuorinen 1990)

If $\Psi(t) = t$, then $f|_{\mathbb{R}^m}$ is Lipschitz, and thus, the measure $\mathcal{H}^m \llcorner f(\mathbb{R}^m)$ is locally finite.

Theorem (Reshetnyak 1994)

If $\Psi(t) = t$, then $f|_{\mathbb{R}^m}$ is C^1 and the quasiplane $f(\mathbb{R}^m)$ is an m-dimensional C^1 embedded submanifold of $\mathbb{R}^n$.

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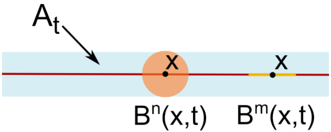

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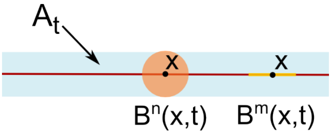

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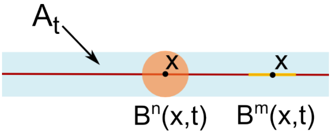

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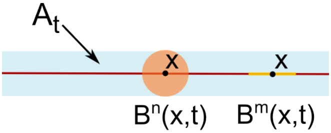

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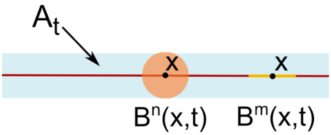

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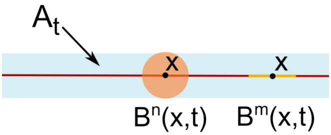

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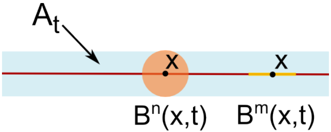

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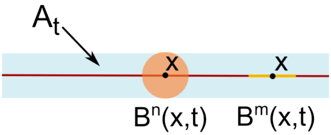

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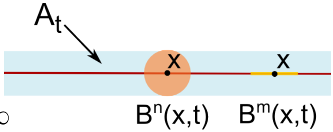

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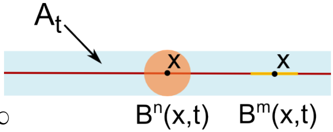

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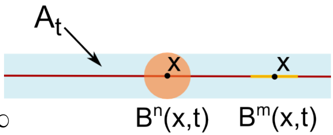

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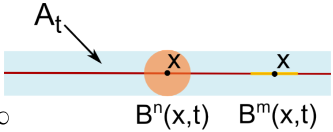

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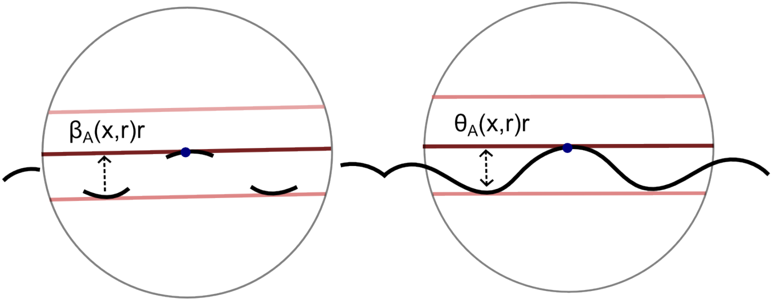
Dilatation Dini Condition	Quasisymmetry Dini Condition	Local Flatness Dini Condition	Bi-Lipschitz Parameterization
$[K_f(A_t)-1]^2$ * $[\log 1/[\ldots]]^2$ →	$\sup_x [H_f(B^n(x,t))-1]^2$ →	$\sup_x \beta_{f(R^m)}(x,t)^2$ →	$f(R^m)$ locally bi-Lipschitz equiv to R^m

Localize Known Global Estimates (BGRT 2012)	$\beta \lesssim H - 1$ Lemma (Prause 2007)	David and Toro's Theorem (2012) Reifenberg's Topological Disk Theorem (1960)
Modulus Estimates	Use Local Hölder Continuity Estimates	θ controlled by $H - 1$ (ABT 2014)

Two Measurements of Flatness of a Set



unilateral flatness: (Jones β-number)

$$\beta_A(x,r) = \frac{1}{r} \inf_{L \in G(n,m)} \sup_{y \in A \cap B(x,r)} \operatorname{dist}(y, x+L)$$

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$$\theta_A(x,r) = \frac{1}{r} \inf_{L \in G(n,m)} \sup_{y \in A \cap B(x,r)} \operatorname{dist}(y, x+L) \vee \sup_{z \in (x+L) \cap B(x,r)} \operatorname{dist}(z, A)$$

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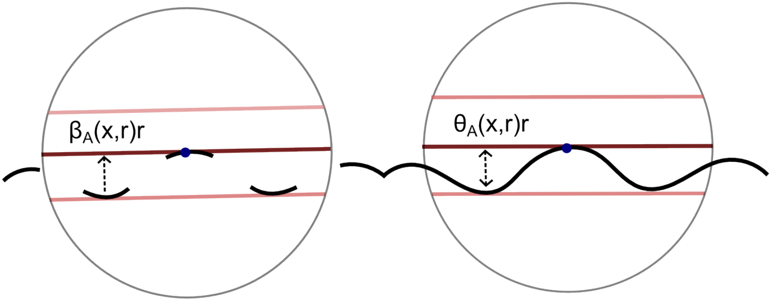

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Bi-Lipschitz Parameterization Theorem

Theorem (David and Toro 2012)

For all $1 \leq m \leq n-1$ *and* $M < \infty$*, there are* $L = L(m,n,M) < \infty$ *and* $\delta_0 = \delta_0(n,m) > 0$ *with the following property. Suppose that* $A \subset \mathbb{R}^n$ *is closed,* $x_0 \in A$*,* $r_0 > 0$*, and* $0 < \delta \leq \delta_0$*.*

If

$$\sup_{x \in A \cap B^n(x_0,r_0)} \int_0^{r_0} \beta_A(x,r)^2 \frac{dr}{r} \leq M < \infty, \qquad (\star)$$

and

$$\theta_A(x,r) \leq \delta \quad \text{for all } x \in A \cap B^n(x_0,r_0) \text{ and } 0 < r \leq r_0, \qquad (\star\star)$$

then there exist (i) an L*-bi-Lipschitz map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *and (ii) an* m*-dimensional plane* V *containing* x_0 *such that*

$$A \cap B^n(x_0, r_0/10) = g(V) \cap B^n(x_0, r_0/10).$$

Quasisymmetry Controls Local Flatness

Lemma (Prause 2007, ABT 2014)

Suppose that $1 \le m \le n-1$, $x \in \mathbb{R}^m$, *and* e *is a unit vector. If* $f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$, *then*

$$\beta_{f(\mathbb{R}^m)}\left(f(x), \frac{1}{2}|f(x+re) - f(x)|\right) \le 144n\left(H_f(B^n(x,2r)) - 1\right).$$

Theorem (Azzam, B., Toro 2014)

For all $\varepsilon > 0$, *there exists* $\eta = \eta(n,m,\varepsilon) > 0$ *such that if* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is quasiconformal,* $H_f(\mathbb{R}^m) \le H$, *and* $H_f(B^n(x_0, 6r_0)) - 1 \le \eta$ *for some* $x_0 \in \mathbb{R}^m$ *and* $r_0 > 0$, *then*

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for all $x \in B^m(x_0, r_0)$ *and* $0 < r \le \frac{1}{54} \operatorname{diam} f(B^m(x_0, r_0))$.

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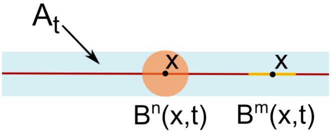

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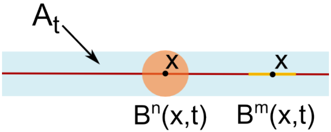

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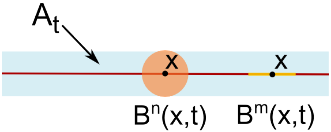

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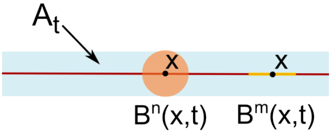

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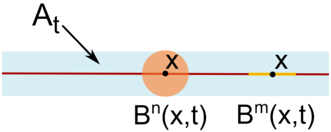
- Restriction to $m \geq 2$: our proof uses a theorem of Gehring that Jh is an A_∞ weight if $h : \mathbb{R}^m \to \mathbb{R}^m$ is quasiconformal. We do not know whether theorem holds when $m = 1$.

Additional Results



Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiconformal.

Theorem (Azzam, B., Toro 2014)

Let $2 \leq m \leq n-1$. If for all $x_0 \in \mathbb{R}^m$ and $t_0 > 0$,

$$\int_{B^m(x_0,t_0)} \int_0^{t_0} (H_f(B^n(x,t)) - 1)^2 \frac{dt}{t} dx \leq C\mathscr{L}^m(B^m(x_0,t_0)),$$

then $f(\mathbb{R}^m)$ has "big pieces of bi-Lipschitz images of $\mathbb{R}^m$".

- BPBI: $\exists L > 1$ and $\alpha > 0$ such that $\forall \xi \in f(\mathbb{R}^m)$ and $s > 0$, $f(\mathbb{R}^m) \cap B^n(\xi, s)$ intersects some L-bi-Lipschitz image of $\mathbb{R}^m$ in a set of $\mathscr{H}^m$ measure at least αs^m.
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