UCONN SEMESTER IN NONSMOOTH ANALYSIS

What is Nonsmooth Analysis?

Matthew Badger and Vasileios Chousionis

University of Connecticut

September 24, 2015

www.math.uconn.edu/nonsmooth

I turn away in fright and horror from this lamentable plague of functions that do not have derivatives.

— C. Hermite, 1893

...clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

— B. Mandelbrot, 1977

I turn away in fright and horror from this lamentable plague of functions that do not have derivatives.

— *C. Hermite, 1893*

...clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

— *B. Mandelbrot, 1977*

Aspects of nonsmooth analysis

1. **nonsmooth objects** (e.g. sets, measures...)
   in (smooth) spaces

2. **nonsmooth functions**
   between (smooth) spaces

3. **nonsmooth spaces**

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$

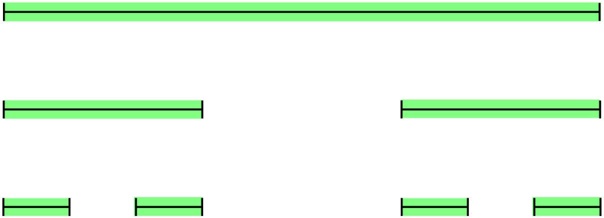
1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k$, $\quad 0 < 2\lambda < 1$
3. Lebesgue measure $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty} (2\lambda)^k = 0$

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$

C(1/3)



1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k, \quad 0 < 2\lambda < 1$
3. Lebesgue measure $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

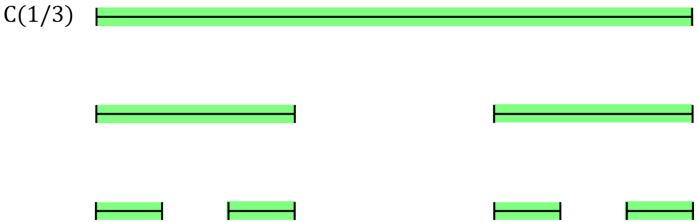
$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$

1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k$, $\quad 0 < 2\lambda < 1$
3. Lebesgue measure $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$



1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k, \quad 0 < 2\lambda < 1$
3. Lebesgue measure $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

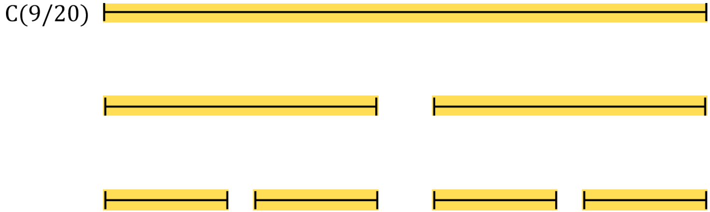
$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$

1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k, \quad 0 < 2\lambda < 1$
3. Lebesgue measure $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

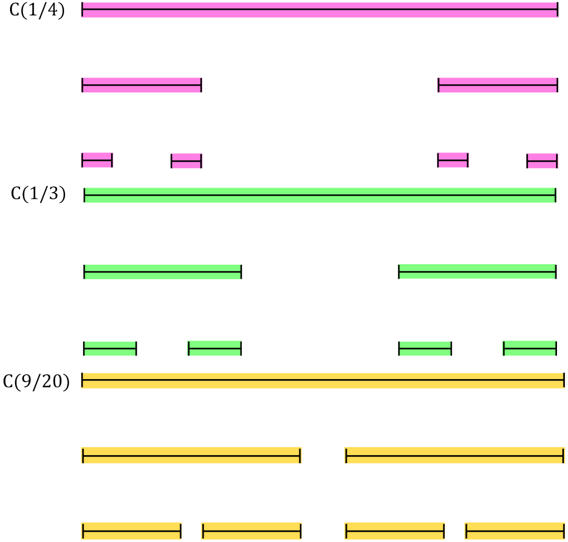
$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$



1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k, \quad 0 < 2\lambda < 1$
3. Lebesgue measure
$\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

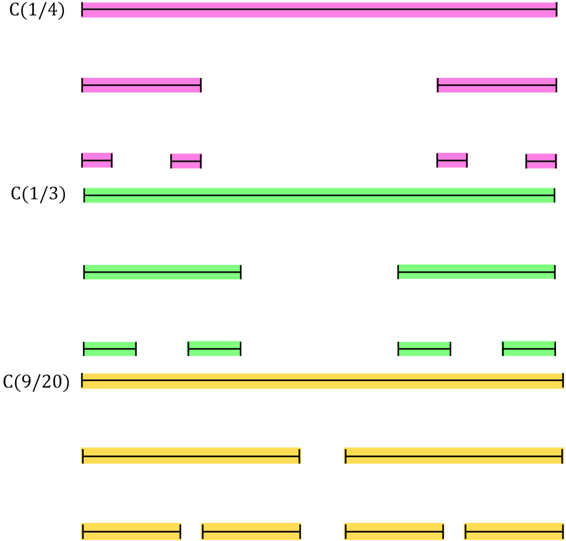
Cantor sets

Lebesgue measure cannot distinguish $C(1/3)$, $C(1/4)$, $C(9/20)$



Cantor sets

But our intuition says that $C(1/4)$ "<" $C(1/3)$ "<" $C(9/20)$



s-dimensional Hausdorff measures $\mathcal{H}^s$ on $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$ is called **s-dimensional Hausdorff measure**

(Borel regular outer measure)

s-dimensional Hausdorff measures $\mathcal{H}^s$ on $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i ; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$ is called ***s*-dimensional Hausdorff measure**

(Borel regular outer measure)

s-dimensional Hausdorff measures $\mathcal{H}^s$ on $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$ is called **s-dimensional Hausdorff measure**

(Borel regular outer measure)

s-dimensional Hausdorff measures $\mathcal{H}^s$ on $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by sets $E_1, E_2, \ldots$ of diameter $\leq \delta$
2. Weight each set in the cover by its diameter to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s : A \subset \bigcup_{i=1}^{\infty} E_i ; \operatorname{diam} E_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

$\mathcal{H}^s$ is called ***s*-dimensional Hausdorff measure**

(Borel regular outer measure)

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x,r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x,r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x,r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x,r)) = 0$
4. Line segments $[a,b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a,b]) = |b-a|$
5. If $s < 1$, then $\mathcal{H}^s([a,b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a,b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x,r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x,r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x,r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x,r)) = 0$
4. Line segments $[a,b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a,b]) = |b-a|$
5. If $s < 1$, then $\mathcal{H}^s([a,b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a,b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

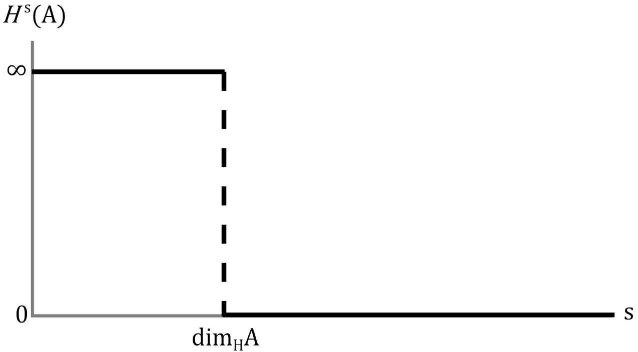
Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$
2. If $s < n$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$
3. If $t > n$, then $\mathcal{H}^t(B_{\mathbb{R}^n}(x, r)) = 0$
4. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$
5. If $s < 1$, then $\mathcal{H}^s([a, b]) = \infty$
6. If $s > 1$, then $\mathcal{H}^s([a, b]) = 0$
7. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) > 0$, then $\mathcal{H}^s(A) = \infty$ for all $s < r$
8. If $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^r(A) < \infty$, then $\mathcal{H}^t(A) = 0$ for all $t > r$

Hausdorff dimension

For any set $A \subseteq \mathbb{R}^n$, there is a unique number $d \in [0, n]$ such that

1. $\mathcal{H}^s(A) = \infty$ for all $s < d$
2. $\mathcal{H}^s(A) = 0$ for all $s > d$

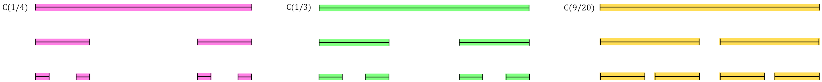


The number $d = \dim_H(A)$ where the transition happens is called the **Hausdorff dimension of A**.

Hausdorff dimension of Cantor sets

For all $\lambda \in (0, 1/2)$, the Cantor set $C(\lambda)$ has Hausdorff dimension

$$\dim_H C(\lambda) = \frac{\log(2)}{\log(1/\lambda)} \in (0, 1)$$



- $\dim_H C(1/4) = \log(2)/\log(4) = 0.5000000\ldots$
- $\dim_H C(1/3) = \log(2)/\log(3) = 0.6309292\ldots$
- $\dim_H C(9/20) = \log(2)/\log(20/9) = 0.8680532\ldots$
- $\dim_H C(\lambda) \downarrow 0$ as $\lambda \downarrow 0$
- $\dim_H C(\lambda) \uparrow 1$ as $\lambda \uparrow 1/2$

Metric spaces

A **metric space** is a set X equipped with a distance function $\text{dist} : X \times X \to [0, \infty)$: for all $x, y, z \in X$

1. nondegenerate: $\text{dist}(x, y) = 0$ if and only if $x = y$
2. symmetric: $\text{dist}(x, y) = \text{dist}(y, x)$
3. triangle inequality: $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.

Metric spaces are core objects / spaces in analysis

The definition of Hausdorff measure and Hausdorff dimension only use the notions of coverings and diameter. So they make sense in any metric space.

Computing the exact Hausdorff measure of a set is very hard.

Computing the Hausdorff dimension of a set is feasible.

Metric spaces

A **metric space** is a set X equipped with a distance function $\text{dist} : X \times X \to [0, \infty)$: for all $x, y, z \in X$

1. nondegenerate: $\text{dist}(x, y) = 0$ if and only if $x = y$
2. symmetric: $\text{dist}(x, y) = \text{dist}(y, x)$
3. triangle inequality: $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.

Metric spaces are core objects / spaces in analysis

The definition of Hausdorff measure and Hausdorff dimension only use the notions of coverings and diameter. So they make sense in any metric space.

Computing the exact Hausdorff measure of a set is very hard.

Computing the Hausdorff dimension of a set is feasible.

Metric spaces

A **metric space** is a set X equipped with a distance function $\mathrm{dist} : X \times X \to [0, \infty)$: for all $x, y, z \in X$

1. nondegenerate: $\mathrm{dist}(x, y) = 0$ if and only if $x = y$
2. symmetric: $\mathrm{dist}(x, y) = \mathrm{dist}(y, x)$
3. triangle inequality: $\mathrm{dist}(x, z) \leq \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$.

Metric spaces are core objects / spaces in analysis

The definition of Hausdorff measure and Hausdorff dimension only use the notions of coverings and diameter. So they make sense in any metric space.

Computing the exact Hausdorff measure of a set is very hard.

Computing the Hausdorff dimension of a set is feasible.

Metric spaces

A **metric space** is a set X equipped with a distance function $\mathrm{dist}: X \times X \to [0, \infty)$: for all $x, y, z \in X$

1. nondegenerate: $\mathrm{dist}(x, y) = 0$ if and only if $x = y$
2. symmetric: $\mathrm{dist}(x, y) = \mathrm{dist}(y, x)$
3. triangle inequality: $\mathrm{dist}(x, z) \leq \mathrm{dist}(x, y) + \mathrm{dist}(y, z)$.

Metric spaces are core objects / spaces in analysis

The definition of Hausdorff measure and Hausdorff dimension only use the notions of coverings and diameter. So they make sense in any metric space.

Computing the exact Hausdorff measure of a set is very hard.

Computing the Hausdorff dimension of a set is feasible.

Metric spaces

A **metric space** is a set X equipped with a distance function $\text{dist} : X \times X \to [0, \infty)$: for all $x, y, z \in X$

1. nondegenerate: $\text{dist}(x, y) = 0$ if and only if $x = y$
2. symmetric: $\text{dist}(x, y) = \text{dist}(y, x)$
3. triangle inequality: $\text{dist}(x, z) \leq \text{dist}(x, y) + \text{dist}(y, z)$.

Metric spaces are core objects / spaces in analysis

The definition of Hausdorff measure and Hausdorff dimension only use the notions of coverings and diameter. So they make sense in any metric space.

Computing the exact Hausdorff measure of a set is very hard.

Computing the Hausdorff dimension of a set is feasible.

Maps between metric spaces

Let X, Y be metric spaces. A map $f : X \to Y$ is **Lipschitz** if there exists $L < \infty$ such that

$$\mathrm{dist}_Y(f(x), f(y)) \leq L\,\mathrm{dist}_X(x, y) \quad \text{for all } x, y \in X.$$

- Natural class of maps between metric spaces: involves only distance functions for the source and target spaces
- Lipschitz maps do not stretch distances "too much" (no more than a bounded multiplicative factor)
- Let $B_X(x, r)$ denote an open ball in X. Let $B_Y(y, s)$ denote an open ball in Y. Then

$$f(B_X(x, r)) \subseteq B_Y(f(x), Lr) \quad \text{for all } x \in X \text{ and } r > 0$$

- For all sets $E \subseteq X$, $\mathrm{diam}\, f(E) \leq L\,\mathrm{diam}\, E$.

Maps between metric spaces

Let X, Y be metric spaces. A map $f : X \to Y$ is **Lipschitz** if there exists $L < \infty$ such that

$$\mathrm{dist}_Y(f(x), f(y)) \leq L\,\mathrm{dist}_X(x, y) \quad \text{for all } x, y \in X.$$

- Natural class of maps between metric spaces: involves only distance functions for the source and target spaces
- Lipschitz maps do not stretch distances "too much" (no more than a bounded multiplicative factor)
- Let $B_X(x, r)$ denote an open ball in X. Let $B_Y(y, s)$ denote an open ball in Y. Then

$$f(B_X(x, r)) \subseteq B_Y(f(x), Lr) \quad \text{for all } x \in X \text{ and } r > 0$$

- For all sets $E \subseteq X$, $\mathrm{diam}\, f(E) \leq L\,\mathrm{diam}\, E$.

Maps between metric spaces

Let X, Y be metric spaces. A map $f : X \to Y$ is **Lipschitz** if there exists $L < \infty$ such that

$$\mathrm{dist}_Y(f(x), f(y)) \le L\,\mathrm{dist}_X(x, y) \quad \text{for all } x, y \in X.$$

- Natural class of maps between metric spaces: involves only distance functions for the source and target spaces
- Lipschitz maps do not stretch distances "too much" (no more than a bounded multiplicative factor)
- Let $B_X(x, r)$ denote an open ball in X. Let $B_Y(y, s)$ denote an open ball in Y. Then

$$f(B_X(x, r)) \subseteq B_Y(f(x), Lr) \quad \text{for all } x \in X \text{ and } r > 0$$

- For all sets $E \subseteq X$, $\mathrm{diam}\, f(E) \le L\,\mathrm{diam}\, E$.

Maps between metric spaces

Let X, Y be metric spaces. A map $f : X \to Y$ is **Lipschitz** if there exists $L < \infty$ such that

$$\mathrm{dist}_Y(f(x), f(y)) \leq L\,\mathrm{dist}_X(x, y) \quad \text{for all } x, y \in X.$$

- Natural class of maps between metric spaces: involves only distance functions for the source and target spaces
- Lipschitz maps do not stretch distances "too much" (no more than a bounded multiplicative factor)
- Let $B_X(x, r)$ denote an open ball in X. Let $B_Y(y, s)$ denote an open ball in Y. Then

$$f(B_X(x, r)) \subseteq B_Y(f(x), Lr) \quad \text{for all } x \in X \text{ and } r > 0$$

- For all sets $E \subseteq X$, $\operatorname{diam} f(E) \leq L \operatorname{diam} E$.

Maps between metric spaces

Let X, Y be metric spaces. A map $f : X \to Y$ is **Lipschitz** if there exists $L < \infty$ such that

$$\mathrm{dist}_Y(f(x), f(y)) \leq L\,\mathrm{dist}_X(x, y) \quad \text{for all } x, y \in X.$$

- Natural class of maps between metric spaces: involves only distance functions for the source and target spaces
- Lipschitz maps do not stretch distances "too much" (no more than a bounded multiplicative factor)
- Let $B_X(x, r)$ denote an open ball in X. Let $B_Y(y, s)$ denote an open ball in Y. Then

$$f(B_X(x, r)) \subseteq B_Y(f(x), Lr) \quad \text{for all } x \in X \text{ and } r > 0$$

- For all sets $E \subseteq X$, $\mathrm{diam}\, f(E) \leq L\,\mathrm{diam}\, E$.

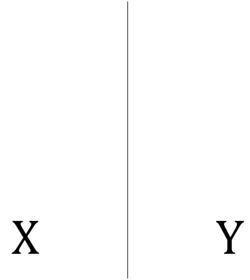
Lipschitz maps don't increase dimension

Theorem

If $f : X \to Y$ is Lipschitz, then $\mathcal{H}^s(f(X)) \leq L^s \mathcal{H}^s(X)$.

Corollary

If $f : X \to Y$ is Lipschitz, then $\dim_H f(X) \leq \dim_H X$.



$$\sum_i (\operatorname{diam} f(E_i))^s \leq L^s \sum_i (\operatorname{diam} E_i)^s.$$

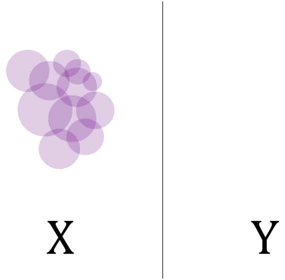
Lipschitz maps don't increase dimension

Theorem

If $f : X \to Y$ is Lipschitz, then $\mathcal{H}^s(f(X)) \le L^s \mathcal{H}^s(X)$.

Corollary

If $f : X \to Y$ is Lipschitz, then $\dim_H f(X) \le \dim_H X$.



$$\sum_i (\operatorname{diam} f(E_i))^s \le L^s \sum_i (\operatorname{diam} E_i)^s.$$

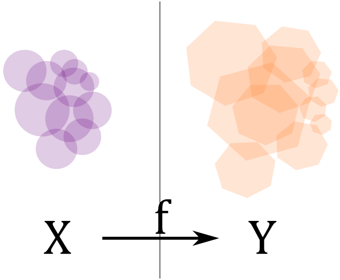
Lipschitz maps don't increase dimension

Theorem

If $f : X \to Y$ is Lipschitz, then $\mathcal{H}^s(f(X)) \leq L^s \mathcal{H}^s(X)$.

Corollary

If $f : X \to Y$ is Lipschitz, then $\dim_H f(X) \leq \dim_H X$.



$$\sum_i (\operatorname{diam} f(E_i))^s \leq L^s \sum_i (\operatorname{diam} E_i)^s.$$

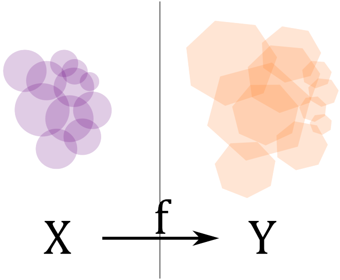
Lipschitz maps don't increase dimension

Theorem

If $f : X \to Y$ is Lipschitz, then $\mathcal{H}^s(f(X)) \le L^s \mathcal{H}^s(X)$.

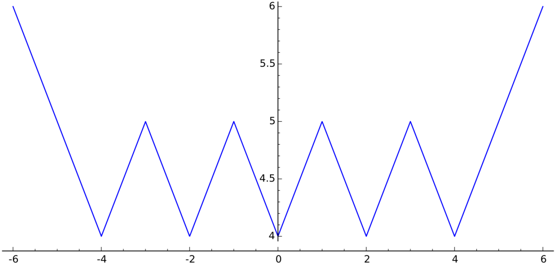
Corollary

If $f : X \to Y$ is Lipschitz, then $\dim_H f(X) \le \dim_H X$.



$$\sum_i (\operatorname{diam} f(E_i))^s \le L^s \sum_i (\operatorname{diam} E_i)^s.$$

How large is the set of points where a Lipschitz function is not differentiable?

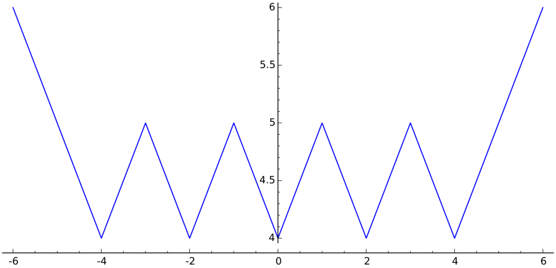


Theorem (Rademacher)

If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then

$$\mathcal{H}^n(\{x \in \mathbb{R}^n : f \text{ is not differentiable at } x\}) = 0.$$

How large is the set of points where a Lipschitz function is not differentiable?



Theorem (Rademacher)

If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then

$$\mathcal{H}^n\left(\{x \in \mathbb{R}^n : f \text{ is not differentiable at } x\}\right) = 0.$$

A research level problem about analysis in metric spaces: Are there versions of Rademacher's theorem in $X \neq \mathbb{R}^n$?

- What does it mean to take a derivative of a function $f : X \to \mathbb{R}$ when X is a metric space?

- When does a metric space have some differentiable structure?

A research level problem about analysis in metric spaces: Are there versions of Rademacher's theorem in $X \neq \mathbb{R}^n$?

- What does it mean to take a derivative of a function $f : X \to \mathbb{R}$ when X is a metric space?

- When does a metric space have some differentiable structure?

A research level problem about analysis in metric spaces: Are there versions of Rademacher's theorem in $X \neq \mathbb{R}^n$?

- What does it mean to take a derivative of a function $f : X \to \mathbb{R}$ when X is a metric space?

- When does a metric space have some differentiable structure?

A research level problem about analysis in metric spaces: Are there versions of Rademacher's theorem in $X \neq \mathbb{R}^n$?

Theorem (Cheeger 1999)

Assume X is a metric space equipped with a measure μ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \quad \textit{for all } x \in X \textit{ and } r > 0$$

and (X,μ) satisfies a Poincaré inequality (Event #3 by Kleiner)

Then there exists

- *positive μ measure sets U_i of dimension $1 \leq n_i < \infty$*
- *Lipschitz maps $\phi_i : U_i \to \mathbb{R}^{n_i}$*

such that for every Lipschitz map $f : X \to \mathbb{R}$, for every $i \geq 1$, and for μ-a.e. $x \in U_i$, there exists $df_x \in \mathbb{R}^{n_i}$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - df_x \cdot (\phi_i(y) - \phi_i(x))|}{\mathrm{dist}(x,y)} = 0.$$

A research level problem about analysis in metric spaces: Are there versions of Rademacher's theorem in $X \neq \mathbb{R}^n$?

Theorem (Cheeger 1999)

Assume X is a metric space equipped with a measure μ such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \quad \textit{for all } x \in X \textit{ and } r > 0$$

and (X,μ) satisfies a Poincaré inequality (Event #3 by Kleiner)
Then there exists

- *positive μ measure sets U_i of dimension $1 \leq n_i < \infty$*
- *Lipschitz maps $\phi_i : U_i \to \mathbb{R}^{n_i}$*

such that for every Lipschitz map $f : X \to \mathbb{R}$, for every $i \geq 1$, and for μ-a.e. $x \in U_i$, there exists $df_x \in \mathbb{R}^{n_i}$ such that

$$\lim_{y \to x} \frac{|f(y) - f(x) - df_x \cdot (\phi_i(y) - \phi_i(x))|}{\mathrm{dist}(x,y)} = 0.$$

Additional classes of nonsmooth maps

- Bi-Lipschitz maps
- Hölder continuous maps
- Sobolev maps
- Quasiconformal maps
- Quasisymmetric maps
- Coarse isometries

The Heisenberg group $\mathbb{H}^n$

- $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$.
- $p = (p_1, \ldots, p_{2n}, p_{2n+1}) := (p', p_{2n+1}) \in \mathbb{H}^n$,
- $p \cdot q = (p_1 + q_1, \cdots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} - A(p', q'))$,

$$A(p', q') = 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i).$$

- $\mathbb{H}^n$ is not Abelian.
- $\|p\|_H = (|p'|^4 + p_{2n+1}^2)^{1/4}$ and $d_H(p,q) = \|p^{-1} \cdot q\|_H$.
- $\delta_r(p) = (rp', r^2 p_{2n+1})$ and $d_H(\delta_r(p), \delta_r(q)) = r d_H(p,q)$.

The Heisenberg group $\mathbb{H}^n$

- $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$.
- $p = (p_1, \ldots, p_{2n}, p_{2n+1}) := (p', p_{2n+1}) \in \mathbb{H}^n$,
- $p \cdot q = (p_1 + q_1, \cdots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} - A(p', q'))$,

$$A(p', q') = 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i).$$

- $\mathbb{H}^n$ is not Abelian.
- $\|p\|_H = (|p'|^4 + p_{2n+1}^2)^{1/4}$ and $d_H(p,q) = \|p^{-1} \cdot q\|_H$.
- $\delta_r(p) = (rp', r^2 p_{2n+1})$ and $d_H(\delta_r(p), \delta_r(q)) = r d_H(p,q)$.

The Heisenberg group $\mathbb{H}^n$

- $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$.
- $p = (p_1, \ldots, p_{2n}, p_{2n+1}) := (p', p_{2n+1}) \in \mathbb{H}^n$,
- $p \cdot q = \left(p_1 + q_1, \cdots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} - A(p', q')\right)$,

$$A(p', q') = 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i).$$

- $\mathbb{H}^n$ is not Abelian.
- $\|p\|_H = (|p'|^4 + p_{2n+1}^2)^{1/4}$ and $d_H(p, q) = \|p^{-1} \cdot q\|_H$.
- $\delta_r(p) = (rp', r^2 p_{2n+1})$ and $d_H(\delta_r(p), \delta_r(q)) = r d_H(p, q)$.

The Heisenberg group $\mathbb{H}^n$

- $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$.
- $p = (p_1, \ldots, p_{2n}, p_{2n+1}) := (p', p_{2n+1}) \in \mathbb{H}^n$,
- $p \cdot q = (p_1 + q_1, \cdots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} - A(p', q'))$,

$$A(p', q') = 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i).$$

- $\mathbb{H}^n$ is not Abelian.
- $\|p\|_H = (|p'|^4 + p_{2n+1}^2)^{1/4}$ and $d_H(p, q) = \|p^{-1} \cdot q\|_H$.
- $\delta_r(p) = (rp', r^2 p_{2n+1})$ and $d_H(\delta_r(p), \delta_r(q)) = r d_H(p, q)$.

The Heisenberg group $\mathbb{H}^n$

- $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$.
- $p = (p_1, \ldots, p_{2n}, p_{2n+1}) := (p', p_{2n+1}) \in \mathbb{H}^n$,
- $p \cdot q = (p_1 + q_1, \cdots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} - A(p', q'))$,

$$A(p', q') = 2 \sum_{i=1}^{n} (p_i q_{i+n} - p_{i+n} q_i).$$

- $\mathbb{H}^n$ is not Abelian.
- $\|p\|_H = (|p'|^4 + p_{2n+1}^2)^{1/4}$ and $d_H(p,q) = \|p^{-1} \cdot q\|_H$.
- $\delta_r(p) = (rp', r^2 p_{2n+1})$ and $d_H(\delta_r(p), \delta_r(q)) = r d_H(p,q)$.

The Heisenberg group $\mathbb{H}^n$

Non-trivial subgroups of $\mathbb{H}^1$ can be:

- Horizontal lines:

$$L_a = \{(t, at, 0) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the (vertical) center of the group

$$T = \{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the vertical subgroups,

$$V_a = \{(t, at, s) \in \mathbb{H}^1 : t, s \in \mathbb{R}\}.$$

The Heisenberg group $\mathbb{H}^n$

Non-trivial subgroups of $\mathbb{H}^1$ can be:

- Horizontal lines:

$$L_a = \{(t, at, 0) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the (vertical) center of the group

$$T = \{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the vertical subgroups,

$$V_a = \{(t, at, s) \in \mathbb{H}^1 : t, s \in \mathbb{R}\}.$$

The Heisenberg group $\mathbb{H}^n$

Non-trivial subgroups of $\mathbb{H}^1$ can be:

- Horizontal lines:

$$L_a = \{(t, at, 0) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the (vertical) center of the group

$$T = \{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the vertical subgroups,

$$V_a = \{(t, at, s) \in \mathbb{H}^1 : t, s \in \mathbb{R}\}.$$

The Heisenberg group $\mathbb{H}^n$

Non-trivial subgroups of $\mathbb{H}^1$ can be:

- Horizontal lines:

$$L_a = \{(t, at, 0) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the (vertical) center of the group

$$T = \{(0, 0, t) \in \mathbb{H}^1 : t \in \mathbb{R}\}.$$

- the vertical subgroups,

$$V_a = \{(t, at, s) \in \mathbb{H}^1 : t, s \in \mathbb{R}\}.$$

The Heisenberg group $\mathbb{H}^n$: some Analysis

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x if there exists $L : \mathbb{R}^n \to \mathbb{R}^m$ linear s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(x-y)|}{|x-y|} = 0.$$

$f : \mathbb{H}^n \to \mathbb{R}^m$ is Pansu-differentiable if there exists an $\mathbb{H}^n$-linear map $L : \mathbb{H}^n \to \mathbb{R}^m$ s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(y^{-1} \cdot x)|}{d_H(x,y)} = 0.$$

Theorem (Pansu Rademacher theorem)

Let $U \subset \mathbb{H}^n$ and $f : U \to \mathbb{R}^m$ Lipshitz. Then f is Pansu differentiable a.e in U.

The Heisenberg group $\mathbb{H}^n$: some Analysis

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x if there exists $L : \mathbb{R}^n \to \mathbb{R}^m$ linear s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(x - y)|}{|x - y|} = 0.$$

$f : \mathbb{H}^n \to \mathbb{R}^m$ is Pansu-differentiable if there exists an $\mathbb{H}^n$-linear map $L : \mathbb{H}^n \to \mathbb{R}^m$ s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(y^{-1} \cdot x)|}{d_H(x, y)} = 0.$$

Theorem (Pansu Rademacher theorem)

Let $U \subset \mathbb{H}^n$ and $f : U \to \mathbb{R}^m$ Lipshitz. Then f is Pansu differentiable a.e in U.

The Heisenberg group $\mathbb{H}^n$: some Analysis

$f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x if there exists $L : \mathbb{R}^n \to \mathbb{R}^m$ linear s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(x - y)|}{|x - y|} = 0.$$

$f : \mathbb{H}^n \to \mathbb{R}^m$ is Pansu-differentiable if there exists an $\mathbb{H}^n$-linear map $L : \mathbb{H}^n \to \mathbb{R}^m$ s.t

$$\lim_{y \to x} \frac{|f(x) - f(y) - L(y^{-1} \cdot x)|}{d_H(x, y)} = 0.$$

Theorem (Pansu Rademacher theorem)

Let $U \subset \mathbb{H}^n$ and $f : U \to \mathbb{R}^m$ Lipshitz. Then f is Pansu differentiable a.e in U.

The Heisenberg group $\mathbb{H}^1$ as a fractal space

- $\mathbb{H}^1 \approx \mathbb{R}^3$.
- $|B(x,r)| = c\, r^4 \implies \dim_H \mathbb{H}^1 = 4$. (!)
- If Σ is a smooth surface then $\dim_H \Sigma = 3$.
- There exist (many) curves γ with $\dim_H \gamma = 2$.

The Heisenberg group $\mathbb{H}^1$ as a fractal space

- $\mathbb{H}^1 \approx \mathbb{R}^3$.
- $|B(x,r)| = c\,r^4 \implies \dim_H \mathbb{H}^1 = 4.$ (!)
- If Σ is a smooth surface then $\dim_H \Sigma = 3$.
- There exist (many) curves γ with $\dim_H \gamma = 2$.

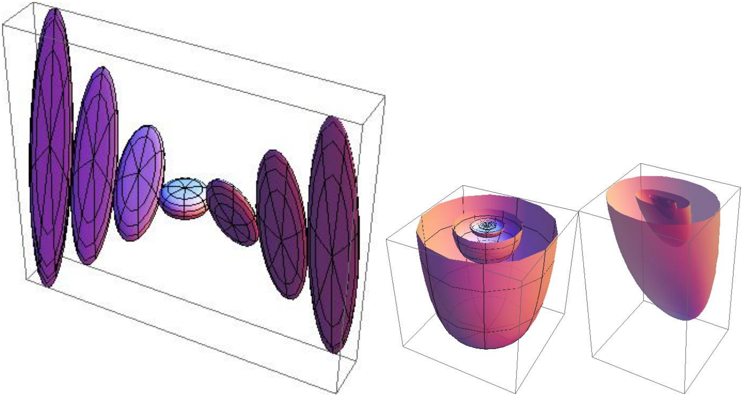
The Heisenberg group $\mathbb{H}^1$ as a fractal space

- $\mathbb{H}^1 \approx \mathbb{R}^3$.
- $|B(x,r)| = c\, r^4 \implies \dim_H \mathbb{H}^1 = 4.$ (!)
- If Σ is a smooth surface then $\dim_H \Sigma = 3$.
- There exist (many) curves γ with $\dim_H \gamma = 2$.

The Heisenberg group $\mathbb{H}^1$ as a fractal space

- $\mathbb{H}^1 \approx \mathbb{R}^3$.
- $|B(x,r)| = c\, r^4 \implies \dim_H \mathbb{H}^1 = 4.$ (!)
- If Σ is a smooth surface then $\dim_H \Sigma = 3$.
- There exist (many) curves γ with $\dim_H \gamma = 2$.

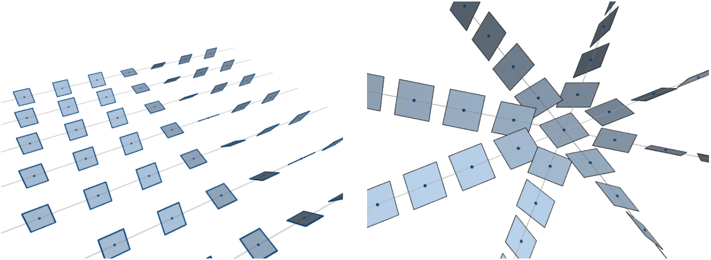
The geometry of $\mathbb{H}$



The geometry of $\mathbb{H}$: Sub-Riemannian structure

Let X_1, X_2 be the left invariant vector fields

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3} \text{ and } X_2 = \partial_{x_2} - 2x_1\partial_{x_3}.$$



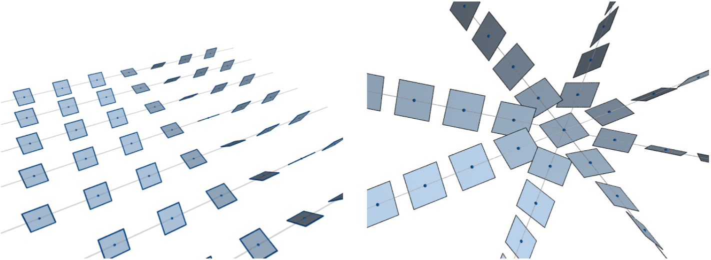
- Motion is only allowed along the horizontal planes:

$$H_p\mathbb{H}^1 = \mathrm{span}\{X_1(p), X_2(p)\}.$$

The geometry of $\mathbb{H}$: Sub-Riemannian structure

Let X_1, X_2 be the left invariant vector fields

$$X_1 = \partial_{x_1} + 2x_2\partial_{x_3} \text{ and } X_2 = \partial_{x_2} - 2x_1\partial_{x_3}.$$



- Motion is only allowed along the horizontal planes:

$$H_p\mathbb{H}^1 = \mathrm{span}\{X_1(p), X_2(p)\}.$$

Horizontal curves in $\mathbb{H}^1$

- Horizontal curve:
  An absolutely continuous curve $\gamma : [0, S] \to \mathbb{H}^1$ such that

$$\dot\gamma(s) \in H_{\gamma(s)}\mathbb{H}^1 \text{ for a.e. } s \in [0, S].$$

- Length of a horizontal curve $\gamma = (x, y, t) : [0, S] \to \mathbb{H}^1$:

$$\ell_H(\gamma) = \int_0^S \sqrt{\dot x(s)^2 + \dot y(s)^2}\, ds = \ell_E(\tilde\gamma)$$

  where $\tilde\gamma = \pi(\gamma) = (x, y) : [0, S] \to \mathbb{R}^2$.

Horizontal curves in $\mathbb{H}^1$

- Horizontal curve:
  An absolutely continuous curve $\gamma : [0, S] \to \mathbb{H}^1$ such that

$$\dot{\gamma}(s) \in H_{\gamma(s)}\mathbb{H}^1 \text{ for a.e. } s \in [0, S].$$

- Length of a horizontal curve $\gamma = (x, y, t) : [0, S] \to \mathbb{H}^1$:

$$\ell_H(\gamma) = \int_0^S \sqrt{\dot{x}(s)^2 + \dot{y}(s)^2} ds = \ell_E(\tilde{\gamma})$$

  where $\tilde{\gamma} = \pi(\gamma) = (x, y) : [0, S] \to \mathbb{R}^2$.

CC distance and geodesics in $\mathbb{H}^1$

- CC-metric in $\mathbb{H}^1$: For $p, q \in \mathbb{H}^1$

  $d_{cc}(p,q)$
  $= \inf\{\ell_H(\gamma) : \gamma : [0,S] \to \mathbb{H}^1 \text{ horizontal }, \gamma(0) = p, \gamma(0) = q\}.$

- d_{cc} is globally equivalent to d_H.
- A *geodesic* between $p, q \in \mathbb{H}^1$ is a horizontal curve of shortest length joining p and q.
- The only geodesically convex subsets of $\mathbb{H}^1$ are the empty set, points, arcs of geodesics and $\mathbb{H}^1$.

The bubble set in $\mathbb{H}^1$

- A horizontal curve γ connecting the origin to $(0, 0, t) \in \mathbb{H}^1$ is a geodesic iff $\tilde{\gamma}$, i.e. its projection on $\mathbb{R}^2$, is a circle.
- Thus there exist infinitely many such geodesics.
- Rotating such a geodesic produces a surface Σ.
- Dilating and translating vertically we obtain sets centered at the origin o:

$$\mathcal{B}(o, R) = \{(p', p_3) \in \mathbb{H}^1 : |p_3| < f_R(|p'|)\}$$

  where $f_R(r) = \frac{1}{4}\left(R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right)$.
- $\partial\mathcal{B}(o, R)$ is C^2 but not C^3.

The bubble set in $\mathbb{H}^1$

- A horizontal curve γ connecting the origin to $(0,0,t) \in \mathbb{H}^1$ is a geodesic iff $\tilde{\gamma}$, i.e. its projection on $\mathbb{R}^2$, is a circle.
- Thus there exist infinitely many such geodesics.
- Rotating such a geodesic produces a surface Σ.
- Dilating and translating vertically we obtain sets centered at the origin o:

$$\mathcal{B}(o,R) = \{(p', p_3) \in \mathbb{H}^1 : |p_3| < f_R(|p'|)\}$$

where $f_R(r) = \frac{1}{4}\left(R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right)$.

- $\partial\mathcal{B}(o,R)$ is C^2 but not C^3.

The bubble set in $\mathbb{H}^1$

- A horizontal curve γ connecting the origin to $(0, 0, t) \in \mathbb{H}^1$ is a geodesic iff $\tilde{\gamma}$, i.e. its projection on $\mathbb{R}^2$, is a circle.
- Thus there exist infinitely many such geodesics.
- Rotating such a geodesic produces a surface Σ.
- Dilating and translating vertically we obtain sets centered at the origin o:

$$\mathcal{B}(o, R) = \{(p', p_3) \in \mathbb{H}^1 : |p_3| < f_R(|p'|)\}$$

where $f_R(r) = \frac{1}{4}\left(R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right)$.

- $\partial\mathcal{B}(o, R)$ is C^2 but not C^3.

The bubble set in $\mathbb{H}^1$

- A horizontal curve γ connecting the origin to $(0, 0, t) \in \mathbb{H}^1$ is a geodesic iff $\tilde{\gamma}$, i.e. its projection on $\mathbb{R}^2$, is a circle.
- Thus there exist infinitely many such geodesics.
- Rotating such a geodesic produces a surface Σ.
- Dilating and translating vertically we obtain sets centered at the origin o:

$$\mathcal{B}(o, R) = \{(p', p_3) \in \mathbb{H}^1 : |p_3| < f_R(|p'|)\}$$

where $f_R(r) = \frac{1}{4}\left(R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right)$.

- $\partial\mathcal{B}(o, R)$ is C^2 but not C^3.

The bubble set in $\mathbb{H}^1$

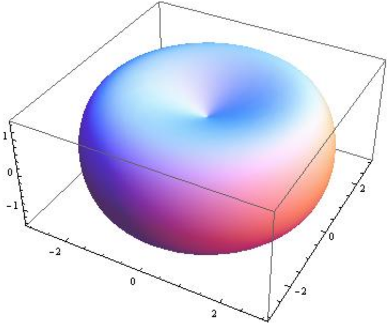
- A horizontal curve γ connecting the origin to $(0,0,t) \in \mathbb{H}^1$ is a geodesic iff $\tilde{\gamma}$, i.e. its projection on $\mathbb{R}^2$, is a circle.
- Thus there exist infinitely many such geodesics.
- Rotating such a geodesic produces a surface Σ.
- Dilating and translating vertically we obtain sets centered at the origin o:

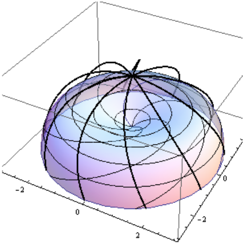
$$\mathcal{B}(o,R) = \{(p', p_3) \in \mathbb{H}^1 : |p_3| < f_R(|p'|)\}$$

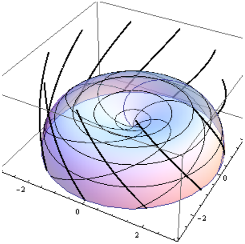
where $f_R(r) = \frac{1}{4}\left(R^2 \arccos\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}\right)$.

- $\partial\mathcal{B}(o,R)$ is C^2 but not C^3.

The bubble set in $\mathbb{H}^1$







The isoperimetric problem in $\mathbb{R}^n$

Isoperimetric Inequality in $\mathbb{R}^n$: For Ω bounded Borel set with finite perimeter measure P.

$$|\Omega|^{\frac{n-1}{n}} \leq C_n P(\Omega)$$

- Sharp constant $C_n = (n^{1-1/n}\omega_{n-1}^{1/n})^{-1}$, where ω_{n-1} = surface area of $\mathbb{S}^{n-1}$.
- Equality holds if and only if Ω is an n-sphere.

The isoperimetric problem in $\mathbb{R}^n$

Isoperimetric Inequality in $\mathbb{R}^n$: For Ω bounded Borel set with finite perimeter measure P.

$$|\Omega|^{\frac{n-1}{n}} \leq C_n P(\Omega)$$

- Sharp constant $C_n = (n^{1-1/n}\omega_{n-1}^{1/n})^{-1}$, where ω_{n-1} = surface area of $\mathbb{S}^{n-1}$.
- Equality holds if and only if Ω is an n-sphere.

The isoperimetric problem in $\mathbb{R}^n$

Isoperimetric Inequality in $\mathbb{R}^n$: For Ω bounded Borel set with finite perimeter measure P.

$$|\Omega|^{\frac{n-1}{n}} \leq C_n P(\Omega)$$

- Sharp constant $C_n = (n^{1-1/n}\omega_{n-1}^{1/n})^{-1}$, where $\omega_{n-1} =$ surface area of $\mathbb{S}^{n-1}$.
- Equality holds if and only if Ω is an n-sphere.

The isoperimetric problem in $\mathbb{R}^n$

Isoperimetric Inequality in $\mathbb{R}^n$: For Ω bounded Borel set with finite perimeter measure P.

$$|\Omega|^{\frac{n-1}{n}} \leq C_n P(\Omega)$$

- Sharp constant $C_n = (n^{1-1/n}\omega_{n-1}^{1/n})^{-1}$, where ω_{n-1} = surface area of $\mathbb{S}^{n-1}$.
- Equality holds if and only if Ω is an n-sphere.

The isoperimetric problem in $\mathbb{H}^1$

In $\mathbb{R}^3$ among all simple closed surfaces with given surface area, the sphere encloses a region of maximal volume.

Theorem (Pansu)

There exists some $C > 0$ such that

$$|\Omega|^{3/4} \leq C P_{\mathbb{H}}(\Omega)$$

for any bounded open set Ω with C^1 boundary.

Conjecture (Pansu 1982)

The best constant in the Heisenberg Isoperimetric inequality is $\frac{3^{3/4}}{4\sqrt{\pi}}$ and equality holds iff Ω is a bubble set.

The isoperimetric problem in $\mathbb{H}^1$

In $\mathbb{R}^3$ among all simple closed surfaces with given surface area, the sphere encloses a region of maximal volume.

Theorem (Pansu)

There exists some $C > 0$ such that

$$|\Omega|^{3/4} \leq C P_{\mathbb{H}}(\Omega)$$

for any bounded open set Ω with C^1 boundary.

Conjecture (Pansu 1982)

The best constant in the Heisenberg Isoperimetric inequality is $\frac{3^{3/4}}{4\sqrt{\pi}}$ and equality holds iff Ω is a bubble set.

The Kakeya conjecture

A Borel set in $\mathbb{R}^n, n \geq 2$ is called a Kakeya, or a Besicovitch set, if it contains a unit segment in every direction and it has zero Lebesgue measure.

- (Besicovitch 20's) Besicovitch sets exist in $\mathbb{R}^n$ for all $n \geq 2$.
- (Davies 1971) In $\mathbb{R}^2$, Besicovitch sets have Hausdorff dimension 2.

Conjecture

If B is a Besicovitch set in $\mathbb{R}^n, n \geq 3$, then

$$\dim_H B = n.$$

The Kakeya conjecture

A Borel set in $\mathbb{R}^n, n \geq 2$ is called a Kakeya, or a Besicovitch set, if it contains a unit segment in every direction and it has zero Lebesgue measure.

- (Besicovitch 20's) Besicovitch sets exist in $\mathbb{R}^n$ for all $n \geq 2$.
- (Davies 1971) In $\mathbb{R}^2$, Besicovitch sets have Hausdorff dimension 2.

Conjecture

If B is a Besicovitch set in $\mathbb{R}^n, n \geq 3$, then

$$\dim_H B = n.$$

The Kakeya conjecture

A Borel set in $\mathbb{R}^n, n \geq 2$ is called a Kakeya, or a Besicovitch set, if it contains a unit segment in every direction and it has zero Lebesgue measure.

- (Besicovitch 20's) Besicovitch sets exist in $\mathbb{R}^n$ for all $n \geq 2$.
- (Davies 1971) In $\mathbb{R}^2$, Besicovitch sets have Hausdorff dimension 2.

Conjecture

If B is a Besicovitch set in $\mathbb{R}^n, n \geq 3$, then

$$\dim_H B = n.$$

The Kakeya conjecture

A Borel set in $\mathbb{R}^n, n \geq 2$ is called a Kakeya, or a Besicovitch set, if it contains a unit segment in every direction and it has zero Lebesgue measure.

- (Besicovitch 20's) Besicovitch sets exist in $\mathbb{R}^n$ for all $n \geq 2$.
- (Davies 1971) In $\mathbb{R}^2$, Besicovitch sets have Hausdorff dimension 2.

Conjecture

If B is a Besicovitch set in $\mathbb{R}^n, n \geq 3$, then

$$\dim_H B = n.$$

Kakeya sets and Harmonic Analysis

- The disk conjecture in $\mathbb{R}^n, n \geq 2$,:
  $\chi_{B(0,1)}$ is an L^p multiplier if $2n/(n+1) < p < 2n/(2n-1)$.
- Fefferman in 1971 disproved the disk conjecture using Besicovitch sets.
- The Restriction Conjecture in $\mathbb{R}^n, n \geq 3$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^\infty(\mathbb{S}^{n-1})} \text{ for } q > 2n/(n-1).$$

- The restriction conjecture is equivalent to the Kakeya conjecture!
- The Kakeya conjecture implies the Bochner-Riesz conjecture.

Kakeya sets and Harmonic Analysis

- The disk conjecture in $\mathbb{R}^n, n \geq 2$,:
  $\chi_{B(0,1)}$ is an L^p multiplier if $2n/(n+1) < p < 2n/(2n-1)$.
- Fefferman in 1971 disproved the disk conjecture using Besicovitch sets.
- The Restriction Conjecture in $\mathbb{R}^n, n \geq 3$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^\infty(\mathbb{S}^{n-1})} \text{ for } q > 2n/(n-1).$$

- The restriction conjecture is equivalent to the Kakeya conjecture!
- The Kakeya conjecture implies the Bochner-Riesz conjecture.

Kakeya sets and Harmonic Analysis

- The disk conjecture in $\mathbb{R}^n, n \geq 2$,:
  $\chi_{B(0,1)}$ is an L^p multiplier if $2n/(n+1) < p < 2n/(2n-1)$.
- Fefferman in 1971 disproved the disk conjecture using Besicovitch sets.
- The Restriction Conjecture in $\mathbb{R}^n, n \geq 3$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^\infty(\mathbb{S}^{n-1})} \text{ for } q > 2n/(n-1).$$

- The restriction conjecture is equivalent to the Kakeya conjecture!
- The Kakeya conjecture implies the Bochner-Riesz conjecture.

Kakeya sets and Harmonic Analysis

- The disk conjecture in $\mathbb{R}^n, n \geq 2,$:
  $\chi_{B(0,1)}$ is an L^p multiplier if $2n/(n+1) < p < 2n/(2n-1)$.
- Fefferman in 1971 disproved the disk conjecture using Besicovitch sets.
- The Restriction Conjecture in $\mathbb{R}^n, n \geq 3$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^\infty(\mathbb{S}^{n-1})} \text{ for } q > 2n/(n-1).$$

- The restriction conjecture is equivalent to the Kakeya conjecture!
- The Kakeya conjecture implies the Bochner-Riesz conjecture.

Kakeya sets and Harmonic Analysis

- The disk conjecture in $\mathbb{R}^n, n \geq 2$,:
  $\chi_{B(0,1)}$ is an L^p multiplier if $2n/(n+1) < p < 2n/(2n-1)$.
- Fefferman in 1971 disproved the disk conjecture using Besicovitch sets.
- The Restriction Conjecture in $\mathbb{R}^n, n \geq 3$,

$$\|\hat{f}\|_{L^q(\mathbb{R}^n)} \lesssim_{n,q} \|f\|_{L^\infty(\mathbb{S}^{n-1})} \text{ for } q > 2n/(n-1).$$

- The restriction conjecture is equivalent to the Kakeya conjecture!
- The Kakeya conjecture implies the Bochner-Riesz conjecture.

Best bounds for the Kakeya conjecture

Let B be a Besicovitch set in $\mathbb{R}^n$

- (Wolff, 95) For $n = 3, 4$,

$$\dim_H B \geq \frac{n+2}{2}.$$

- (Katz and Tao, 02) For $n \geq 5$,

$$\dim_H B \geq (2 - \sqrt{2})(n - 4) + 3.$$

- Although the best known bounds are not due to him, Bourgain has greatly contributed in the progress of the problem.

Best bounds for the Kakeya conjecture

Let B be a Besicovitch set in $\mathbb{R}^n$

- (Wolff, 95) For $n = 3, 4$,

$$\dim_H B \geq \frac{n+2}{2}.$$

- (Katz and Tao, 02) For $n \geq 5$,

$$\dim_H B \geq (2 - \sqrt{2})(n-4) + 3.$$

- Although the best known bounds are not due to him, Bourgain has greatly contributed in the progress of the problem.

Best bounds for the Kakeya conjecture

Let B be a Besicovitch set in $\mathbb{R}^n$

- (Wolff, 95) For $n = 3, 4$,

$$\dim_H B \geq \frac{n+2}{2}.$$

- (Katz and Tao, 02) For $n \geq 5$,

$$\dim_H B \geq (2 - \sqrt{2})(n - 4) + 3.$$

- Although the best known bounds are not due to him, Bourgain has greatly contributed in the progress of the problem.

Best bounds for the Kakeya conjecture

Let B be a Besicovitch set in $\mathbb{R}^n$

- (Wolff, 95) For $n = 3, 4$,

$$\dim_H B \geq \frac{n+2}{2}.$$

- (Katz and Tao, 02) For $n \geq 5$,

$$\dim_H B \geq (2 - \sqrt{2})(n-4) + 3.$$

- Although the best known bounds are not due to him, Bourgain has greatly contributed in the progress of the problem.

UCONN SEMESTER IN NONSMOOTH ANALYSIS

Public University Lecture:

Michael Barnsley

(Australian National University)

November 17

COLLOQUIUM LECTURES

October 1	Fabrice Baudoin (Purdue)
October 8	Bruce Kleiner (NYU)
October 22	Pertti Mattila (Helsinki)
October 29	Nages Shanmugalingam (Cincinatti)
November 12	Luca Capogna (Worcester)
November 19	Xavier Tolsa (Barcelona)

Organizing committee

Matthew Badger
Vasileios Chousionis
Masha Gordina
Luke Rogers
Alexander Teplyaev

For more information visit:
www.math.uconn.edu/nonsmooth

UCONN
UNIVERSITY OF CONNECTICUT