Nodal Domains of Homogeneous Caloric Polynomials

Joint work with
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University of Arkansas

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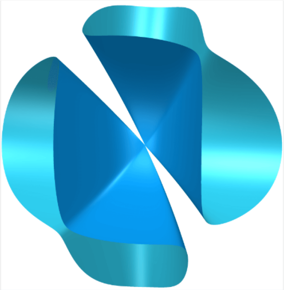
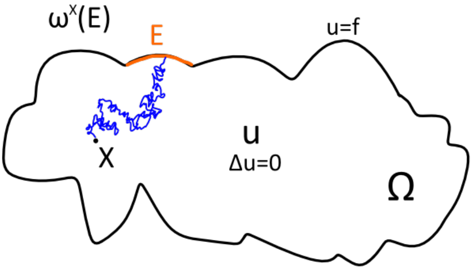


Research partially supported by NSF DMS 2154047

Part 1 – Motivation

Dirichlet Problem and Harmonic Measure

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a regular domain for (D).



Dirichlet Problem

Given $f \in C_c(\partial\Omega)$,
find $u \in C^2(\Omega) \cap C(\overline{\Omega})$:

$$(D) \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

$$\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \cdots + \partial_{x_n x_n}$$

$\exists!$ family of probability measures $\{\omega^X\}_{X\in\Omega}$ on the boundary $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

$$u(X) = \int_{\partial\Omega} f(Q)d\omega^X(Q) \qquad \text{solves (D)}$$

For unbounded domains, we may also consider harmonic measure with pole at infinity.

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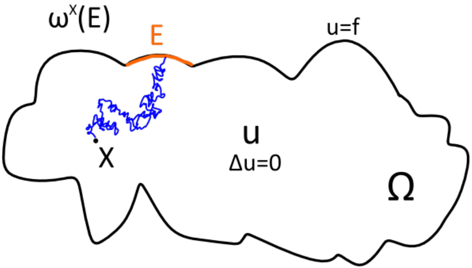

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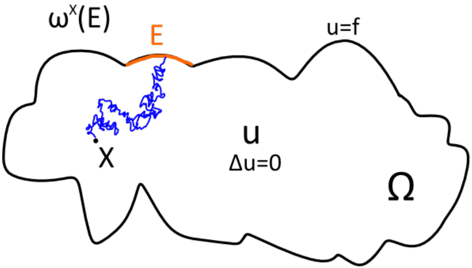

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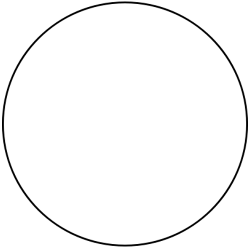
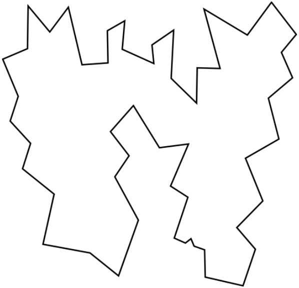
Examples of Regular Domains

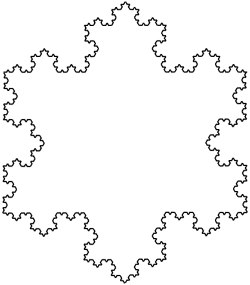
NTA domains introduced by Jerison and Kenig 1982:
Quantitative Openness + Quantitative Path Connectedness



Smooth Domains



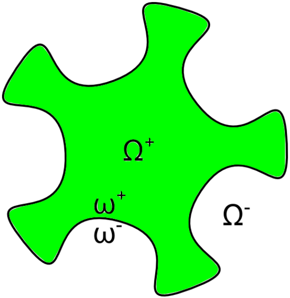
Lipschitz Domains



Quasispheres

(e.g. snowflake)

Two-Phase Free Boundary Regularity Problem



$\Omega \subset \mathbb{R}^n$ is a **2-sided domain** if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial\Omega^+ = \partial\Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with harmonic measures ω^+ on Ω^+ and ω^- on Ω^-.

If $\omega^+ \ll \omega^- \ll \omega^+$, then $f = \dfrac{d\omega^-}{d\omega^+}$ exists, $f \in L^1(d\omega^+)$.

Determine the extent to which existence or regularity of f controls the geometry or regularity of the boundary $\partial\Omega$.

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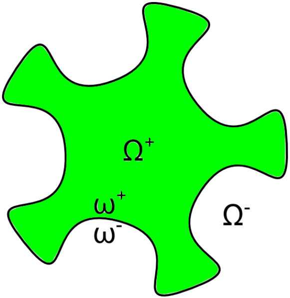

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Regularity of a boundary can be expressed in terms of geometric blowups of the boundary

Measure-Theoretic Tangents Exist at Typical Points

Theorem (Azzam-Mourgoglou-Tolsa-Volberg 2016)

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain with harmonic measures $\omega^\pm$ on $\Omega^\pm$. If $\omega^+ \ll \omega^- \ll \omega^+$, then $\partial\Omega = G \cup N$, where

1. $\omega^\pm(N) = 0$ and $\mathcal{H}^{n-1} \llcorner G$ is locally finite,
2. $\omega^\pm \llcorner G \ll \mathcal{H}^{n-1} \llcorner G \ll \omega^\pm \llcorner G$,
3. up to a $\omega^\pm$-null set, G is contained in a countable union of graphs of Lipschitz functions $f_i : V_i \to V_i^\perp$, $V \in G(n, n-1)$.

In contemporary Geometric Measure Theory, we express (3) by saying $\omega^\pm$ are $(n-1)$-dimensional **Lipschitz graph rectifiable**.

In particular, if $\omega^+ \ll \omega^- \ll \omega^+$, then at $\omega^\pm$-a.e. $x \in \partial\Omega$, there is a unique $\omega^\pm$-approximate tangent plane $V \in G(n, n-1)$:

$$\limsup_{r\downarrow 0} \frac{\omega^\pm(B(x,r))}{r^{n-1}} > 0 \quad \text{and} \quad \limsup_{r\downarrow 0} \frac{\omega^\pm(B(x,r) \setminus \mathrm{Cone}(x+V,\alpha))}{r^{n-1}} = 0$$

for every cone around the $(n-1)$-plane $x + V$.

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Example: Polynomial Singularity

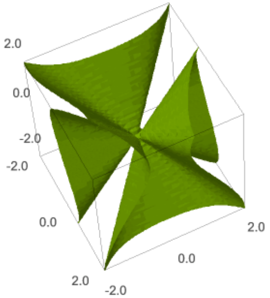

Figure: The zero set of Szulkin's (1978) degree 3 homogeneous harmonic polynomial $p(x, y, z) = x^3 - 3xy^2 + z^3 - 1.5(x^2 + y^2)z$

$\Omega^{\pm} = \{p^{\pm} > 0\}$ is a 2-sided NTA domain, $\omega^+ = \omega^-$ (pole at infinity), $\log \frac{d\omega^-}{d\omega^+} \equiv 0$ but $\partial\Omega^{\pm} = \{p = 0\}$ is not smooth at the origin.

$\log \frac{d\omega^-}{d\omega^+}$ is smooth $\not\Rightarrow$ $\partial\Omega$ is smooth

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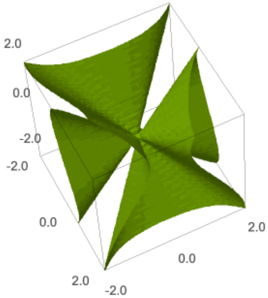

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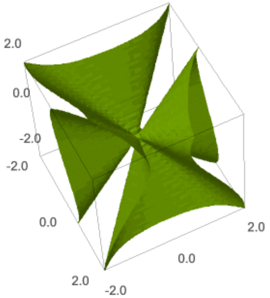

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Useful Terminology: Local Set Approximation (B-Lewis 2015)

Let $A \subset \mathbb{R}^n$ be closed, let $x_i \in A$, let $x_i \to x \in A$, and let $r_i \downarrow 0$.

If $\dfrac{A-x}{r_i} \to T$, we say that T is a **tangent set** of A at x.

- Attouch-Wets topology: $\Sigma_i \to \Sigma$ if and only if for every $r > 0$, $\lim_{i\to\infty}\left(\sup_{x\in\Sigma_i\cap B_r}\operatorname{dist}(x,\Sigma) + \sup_{y\in\Sigma\cap B_r}\operatorname{dist}(y,\Sigma_i)\right) = 0$
- There is at least one tangent set at each $x \in A$.
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If $\dfrac{A-x_i}{r_i} \to S$, we say that S is a **pseudotangent set** of A at x.

- Every tangent set of A at x is a pseudotangent set of A at x.
- There could be pseudotangent sets that are not tangent sets.

We say that A is **locally bilaterally well approximated by** $\mathcal{S}$ if every pseudotangent set of A belongs to $\overline{\mathcal{S}}$.

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Tangents under Weak Regularity

Theorem (Kenig-Toro 2006, B 2011, B-Engelstein-Toro 2017)

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain equipped with harmonic measures $\omega^\pm$ on $\Omega^\pm$. If $\omega^+ \ll \omega^- \ll \omega^+$ and $f = \dfrac{d\omega^-}{d\omega^+}$ has $\log f \in \mathrm{VMO}(d\omega^+)$, then

- $\partial\Omega$ is locally bilaterally well approximated by zero sets of harmonic polynomials $p : \mathbb{R}^n \to \mathbb{R}$ of degree at most d_0 such that $\Omega_p^\pm = \{x : \pm p(x) > 0\}$ are NTA domains and $\dim_M \partial\Omega = n - 1$.

Moreover, we can partition $\partial\Omega = \Gamma_1 \cup S = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$.

- Γ_1 is relatively open in $\partial\Omega$, Γ_1 is locally bilaterally well approximated by $(n-1)$-planes, and $\dim_M \Gamma_1 = n - 1$
- S is closed, $\omega^\pm(S) = 0$ and $\dim_M S \leq n - 3$
- $S = \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$, where $x \in \Gamma_d \Leftrightarrow$ every tangent set of $\partial\Omega$ at x is the zero set of a homogeneous harmonic polynomial q of degree d such that $\Omega_q^\pm$ are NTA domains.

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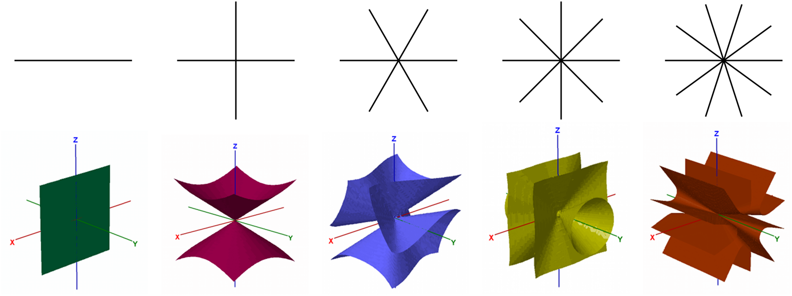
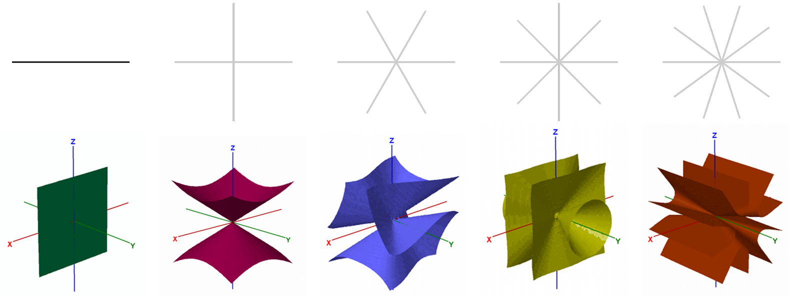


Zero Sets of HHP in $\mathbb{R}^2$ and $\mathbb{R}^3$ of Degrees 1, 2, 3, 4, 5

Admissible Tangents

- In first row, only the first example (degree 1) separates the plane into 2-sided NTA domains
- In second row, only the first, third, and fifth examples (odd degrees) separate space into 2-sided NTA domains (Lewy 1977)
- In $\mathbb{R}^4$ or higher dimensions, there are examples of all degrees that separate space into 2-sided NTA domains (B-Engelstein-Toro 2017)



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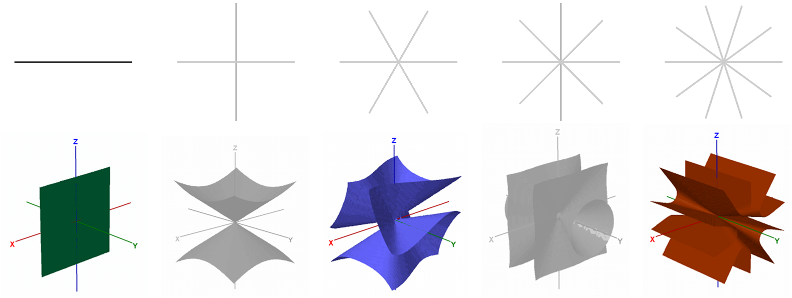

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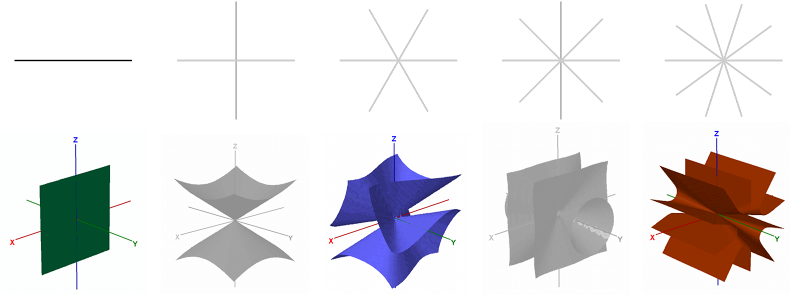

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Theorem (Engelstein 2016 + B-Engelstein-Toro 2020)

Assume that $\Omega^{\pm}$ are NTA and $\log f \in C^{0,\alpha}(\partial\Omega)$ (Hölder continuous).
Then $\partial\Omega$ has a **unique tangent set** at every $x \in \partial\Omega$.

Theorem (B-Engelstein-Toro 2023)

For any $d \geq 1$, there exist examples where $\Omega^{\pm}$ are NTA, $\Gamma_d \neq \emptyset$, $\log f \in C(\partial\Omega) \setminus \bigcup_{\alpha>0} C^{0,\alpha}(\partial\Omega)$ (continuous, but not Hölder) and $\partial\Omega$ has **continuum of distinct tangent sets** at some $x \in \Gamma_d$.

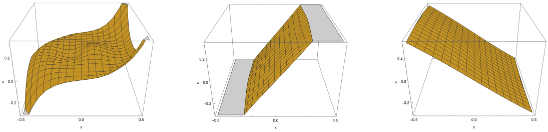
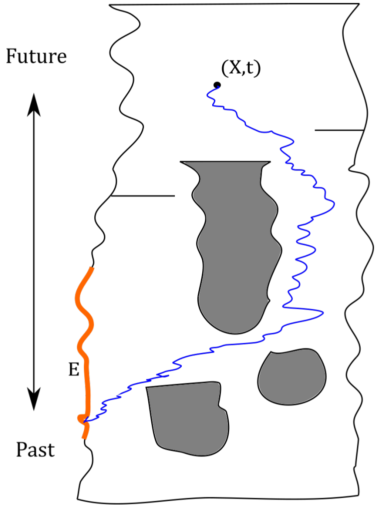


Figure: Blow-ups Σ/r of the interface $\Sigma = \partial\Omega^{\pm}$ of the graph domains associated to $v(x,y) = x \log|\log(\sqrt{x^2+y^2})| \sin(\log|\log(\sqrt{x^2+y^2})|)$ at a flat point $0 \in \Gamma_1$. **Left:** $r = 1$ **Center:** $r = 10^{-6}$ **Right:** $r = 10^{-12}$

We want to carry out the same sort of investigation in the context of the heat equation

Heat Dirichlet Problem and Caloric Measure

Let $n \geq 1$ and let $\Omega \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ be a regular domain for (HD).
The **essential boundary** $\partial_e\Omega$ includes the part of $\partial\Omega$ that is accessible by paths in Ω moving backwards-in-time.



$\exists!$ family of probability measures $\{\omega^{X,t}\}_{(X,t)\in\Omega}$ on $\partial_e\Omega$ called **caloric measure** of Ω with pole at $(X,t) \in \Omega$ such that

$$u(X,t) = \int_{\partial_e\Omega} f(Y,s)\,d\omega^{X,t}(Y,s)$$

solves **heat Dirichlet problem** with boundary data $f \in C_c(\partial_e\Omega)$: $u \in C^2(\Omega)$, $\partial_t u - \Delta_X u = 0$ in Ω and $u \overset{*}{=} f$ on $\partial_e\Omega$

*requires interpretation on $\partial_{ss}\Omega \subset \partial_e\Omega$

Theorem

Caloric measure of present & future is zero: $\omega^{X,t}(\{(Y,s) \in \partial\Omega : s \geq t\}) = 0$

Two-Phase Caloric Free Boundary Regularity

Caloric Analogue of Kenig-Toro (2006) + B (2011):

Theorem (Mourgoglou-Puliatti 2021)

Let $\Omega^+ = \mathbb{R}^{n+1} \setminus \overline{\Omega^-}$ and $\Omega^- = \mathbb{R}^{n+1} \setminus \overline{\Omega^+}$ be complimentary domains with "nice" (for heat potential theory) common boundary.
Let $\omega^\pm$ be caloric measures on $\Omega^\pm$ with poles at $(X_0^\pm, t_0)$.

If $\omega^+ \ll \omega^- \ll \omega^+$ and $f = \frac{d\omega^-}{d\omega^+}$ has $\log f \in \mathrm{VMO}(d\omega^+)$, then

- $\partial\Omega$ is locally bilaterally well approximated[1] by zero sets of caloric polynomials $p : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree at most d_0 such that $\Omega_p^\pm = \{x : \pm p(x) > 0\}$ are connected.

Moreover, we can partition $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{d_0}$ where

- $(X, t) \in \Gamma_d \Leftrightarrow$ every tangent set of $\partial\Omega$ at (X, t) is the zero set of a parabolically homogeneous caloric polynomial q of degree d such that $\Omega_q^\pm$ are connected.

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Part 2 – Nodal Domains of Caloric Polynomials

Nodal Domains

Let $u : \mathbb{R}^{n+1} \to \mathbb{R}$ be continuous.

- The **nodal set** of u is $\{u = 0\}$ (possibly empty).
- A **nodal domain** of u is a connected component of $\{u \neq 0\}$.

$\mathcal{N}(u) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ denotes number of nodal domains of u

A polynomial $p : \mathbb{R}^{n+1} \to \mathbb{R}$ in variables $(X, t) = (X_1, \dots, X_n, t)$ with real coefficients is **caloric** if p solves heat equation: $\partial_t p - \Delta_X p \equiv 0$.

Lemma

If $p : \mathbb{R}^{n+1} \to \mathbb{R}$ is a non-constant caloric polynomial of degree d, then $2 \leq \mathcal{N}(p) \leq (d+1)^n(d+2)$.

Proof: There is a point $(X, t) \in \mathbb{R}^{n+1}$ with $t < 0$ at which $p(X, t) = 0$ (orthogonality of negative-time slices). Applying the mean value property for heat balls, there are $(X^{\pm}, t^{\pm})$ near (X, t) with $t^{\pm} < t$ at which $\pm p(X^{\pm}, t^{\pm}) > 0$. Hence $\mathcal{N}(p) \geq 2$. The upper bound holds for arbitrary polynomials in $\mathbb{R}^{n+1}$ of degree d by Milnor (1964).

Time Coefficients of Caloric Polynomials

Suppose that we have a polynomial solution of the heat equation:

$$p(X,t) = t^d p_d(X) + t^{d-1} p_{d-1}(X) + \cdots + t p_1(X) + p_0(X), \quad p_d(X) \not\equiv 0$$

Applying the heat opeartor $\partial_t - \Delta_X$ we get:

$$t^d(-\Delta_X p_d(X)) + t^{d-1}(d p_d(X) - \Delta_X p_{d-1}(X)) + \cdots + t(2p_2(X) - \Delta_X p_1(X)) + (p_1(X) - \Delta_X p_0(X)) = 0$$

$\Delta_X p_d(X) = 0$:	$p_d(X)$ is harmonic
$\Delta_X p_{d-1}(X) = d p_d(X)$:	$p_{d-1}(X)$ is bi-harmonic
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Homogeneous Caloric Polynomials

If $u(X, t)$ solves the heat equation $\lambda > 0$, then $v(X, t) \equiv u(\lambda X, \lambda^2 t)$ solves the heat equation.

A **homogeneous caloric polynomial (HCP) of degree** d is a caloric polynomial p on $\mathbb{R}^{n+1}$ that is **parabolically homogeneous**:

$$p(\lambda X, \lambda^2 t) \equiv \lambda^d p(X, t)$$

- For any exponent $k \in \mathbb{N}$ and multi-index $\alpha \in \mathbb{N}^n$, the monomial $t^k X^\alpha$ is parabolically homogeneous of degree $2k + |\alpha|$.
- Parabolic and algebraic homogeneity are distinct notions:

  $$p(x, t) = t^2 + tx^2 + x^4/12$$

  is an HCP of degree 4 in $\mathbb{R}^{1+1}$, but p is not algebraically homogeneous. Nevertheless, the parabolic degree and the algebraic degree of an HCP always coincide.
- Any time-dependent HCP has degree at least 2.

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HCP in $\mathbb{R}^{1+1}$

Fact: In $\mathbb{R}^1$, harmonic functions linear: $u''(x) = 0 \Rightarrow u(x) = mx + b$.

Corollary: Up to scaling by a constant, there is a unique HCP in $\mathbb{R}^{1+1}$ of each degree $d \geq 1$: $p_0(x,t) = 1$ $p_1(x,t) = x$

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$$x^2 - y^2 = 2(t + \tfrac{1}{2}x^2) - 2(t + \tfrac{1}{2}y^2) = 2p_2(x,t) - 2p_2(y,t)$$

HCP in $\mathbb{R}^{1+1}$

Fact: In $\mathbb{R}^1$, harmonic functions linear: $u''(x) = 0 \Rightarrow u(x) = mx + b$.

Corollary: Up to scaling by a constant, there is a unique HCP in $\mathbb{R}^{1+1}$ of each degree $d \geq 1$: $p_0(x,t) = 1$ $p_1(x,t) = x$

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$p_{2k}(x,t) = t^k + \frac{k}{2!}t^{k-1}x^2 + \frac{k(k-1)}{4!}t^{k-2}x^4 + \cdots + \frac{k!}{(2k)!}x^{2k}$

$p_{2k+1}(x,t) = t^k x + \frac{k}{3!}t^{k-1}x^3 + \frac{k(k-1)}{5!}t^{k-2}x^5 + \cdots + \frac{k!}{(2k+1)!}x^{2k+1}$

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Theorem (Factorization Lemma)

For all $d \geq 2$, the "basic hcp" $p_d(x,t)$ in $\mathbb{R}^{1+1}$ assumes the form

$$p_d(x,t) = \begin{cases} (t + a_{d,1}x^2)\cdots(t + a_{d,k}x^2) & \text{when } d = 2k \text{ is even,} \\ x(t + a_{d,1}x^2)\cdots(t + a_{d,k}x^2) & \text{when } d = 2k+1 \text{ is odd,} \end{cases}$$

for some distinct numbers $0 < a_{d,1} < \cdots < a_{d,k}$. Moreover,

$$p_{2k-1}(x,t) = x(t + a_1x^2)\cdots(t + a_{k-1}x^2), \quad p_{2k+1}(x,t) = x(t + c_1x^2)\cdots(t + c_kx^2),$$

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with the a_i's, b_i's, and c_i's each listed in increasing order, then the coefficients associated with consecutive polynomials are interlaced:

$$\begin{cases} b_1 < a_1 < b_2 < a_2 < \cdots < a_{k-1} < b_k, \\ c_1 < b_1 < c_2 < b_2 < \cdots < b_{k-1} < c_k < b_k. \end{cases}$$

Why? $p_d(x,-1) = \frac{\lceil d/2 \rceil !}{d!} H_d(x/2)$, where $H_d(x)$ is the so-called **Hermite orthogonal polynomial**. Use facts about these and parabolic scaling.

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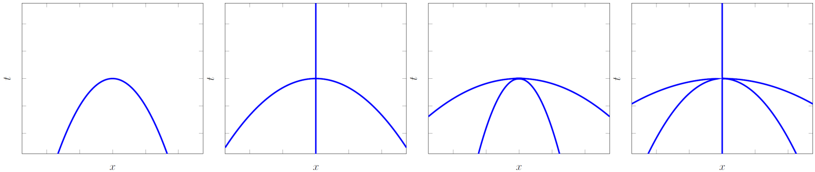


- The nodal set of a degree d hcp in $\mathbb{R}^{1+1}$ is a union of $\lfloor d/2 \rfloor$ nested, downward-opening parabolas with a common turning point at the origin, and when d is odd, an additional vertical line (the t-axis).
- From left to right, we illustrate the cases $d = 2$, ..., $d = 5$.
- Inside the nodal set of $p_d p_{d+1}$, the "nodal parabolas" of consecutive hcps p_d and p_{d+1} are intertwined:
  - "widest" parabola of p_{d+1} sits above "widest" parabola of p_d;
  - "widest" parabola of p_d above "second widest" parabola of p_{d+1};
  - and so on...

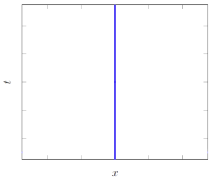
Corollary

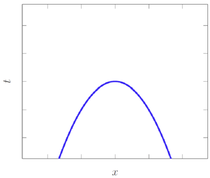
Any hcp $p(x, t)$ in $\mathbb{R}^{1+1}$ of degree $d \geq 1$ has exactly $2\lceil d/2 \rceil$ nodal domains.

Consequence:

In the $n = 1$ case of Mourgoglou and Puliatti's theorem, $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+)$ implies that $\partial\Omega = \Gamma_1 \cup \Gamma_2$.

The only tangent sets of the boundary are:





Remark: So far this improvement only uses classical results

Part 3 – New Results

Minimum and Maximum Number of Nodal Domains

Let $m_{n,d}$ and $M_{n,d}$ denote the minimum and maximum number of nodal domains among time-dependent HCP in $\mathbb{R}^{n+1}$ of degree d. Recall that $m_{1,d} = M_{1,d} = 2\lceil d/2 \rceil$.

Theorem (B-Jeznach 2024: minimum number $m_{n,d}$)

When $n = 2$,

$$m_{2,d} = \begin{cases} 2, & \text{when } d \not\equiv 0 \pmod 4, \\ 3, & \text{when } d \equiv 0 \pmod 4. \end{cases}$$

When $n \geq 3$, we have $m_{n,d} = 2$ for all $d \geq 2$.

Theorem (B-Jeznach 2024: maximum number $M_{n,d}$)

For all $n \geq 2$, $M_{n,d} = \Theta(d^n)$ as $d \to \infty$. More precisely,

$$\left\lfloor \frac{d}{n} \right\rfloor^n \leq M_{n,d} \leq \binom{n+d}{n} \quad \text{for all } n \geq 2,\ d \geq 2.$$

The method of proof is constructive and gives examples acheiving $m_{n,d}$.

Example 1: The polynomial

$$p(x, y, t) = 150t(3x + y) + 27x^3 + 267x^2y + 144xy^2 - 64y^3$$

is an hcp of degree 3 in $\mathbb{R}^{2+1}$ and $\mathcal{N}(p) = 2$.

Example 2: The polynomial

$$p(x, y, t) = 7500t^2 + 150t(37x^2 - 7xy + 13y^2) + 192x^4 + 176x^3y + 1623x^2y^2 - 351xy^3 - 108y^4$$

is an hcp of degree 4 in $\mathbb{R}^{2+1}$ and $\mathcal{N}(p) = 3$.

Example 3: The polynomial

$$p(x, y, z, t) = 12t^2 + 12tx^2 + x^4 + y^4 - 6y^2z^2 + z^4$$

is an hcp of degree 4 in $\mathbb{R}^{3+1}$ and $\mathcal{N}(p) = 2$.

The zero set in each example is smooth outside of the origin.

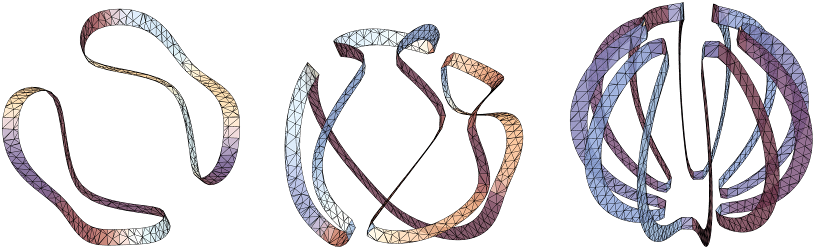


Figure: Gallery of nodal sets of homogeneous caloric polynomials in $\mathbb{R}^{2+1}$ achieving the minimum number $m_{2,d}$ of nodal domains

From left to right, $d = 4$, $d = 5$, and $d = 6$

For increased visibility, we show the intersection of the full nodal set with a spherical annulus

Consequence:

Corollary

Let $\Omega^{\pm} \subset \mathbb{R}^{n+1}$ be as in Mourgoglou and Puliatti's theorem.

Assume that $\log \frac{d\omega^-}{d\omega^+} \in \mathrm{VMO}(d\omega^+)$

When $n = 2$,

$$\partial\Omega = \bigcup_{k \geq 0} \Gamma_{4k+1} \cup \Gamma_{4k+2} \cup \Gamma_{4k+3};$$

for every $d \not\equiv 0 \pmod 4$, the stratum Γ_d is nonempty for some pair of domains satisfying the free boundary condition.

When $n = 3$, the stratum Γ_d can be nonempty for every $d \geq 1$.

Part 4 — Some Proof Ideas

Let's focus on the problem of finding HCP in $\mathbb{R}^{2+1}$ that realize the minimal number of nodal domains.

Counting the nodal domains of an HCP p is equivalent to counting the nodal domains of $p|_{\mathbb{S}^2}$. We can attempt to implement Lewy's method for spherical harmonics (1977) in the parabolic context:

1. Begin with an HCP ϕ_1 of degree d whose nodal set can be **described explicitly**.
2. Find another HCP ϕ_2 of degree d so that the nodal set of the perturbation $u = \phi_1 - \epsilon\phi_2$ in $\mathbb{S}^2$ is a single Jordan curve.

The key difficulty in this strategy is finding certain compatibility conditions between ϕ_1, ϕ_2.

Lemma (Lewy 1977, B-Jeznach 2024)

Suppose that $G : B_r(0) \subset \mathbb{R}^2 \to \mathbb{R}$ takes the form of a product $G(x, y) = \prod_{i=1}^m g_i(x, y)$ for some $m \geq 2$, where $g_1, \ldots, g_m : B_r(0) \to \mathbb{R}$ are real-analytic functions satisfying

- $g_i(0, 0) = 0$ and $\partial_y g_i(0, 0) \neq 0$ for all i,
- $\{g_i = 0\} \cap \{g_j = 0\} = \{(0, 0)\}$ for all $i \neq j$.

If $F : B_r(0) \to \mathbb{R}$ is C^1 and $F(0, 0) > 0$, then there exists $\tau \in (0, r)$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the nodal set of $G - \epsilon F$ in $B_\tau(0)$ consists of m pairwise disjoint simple curves, one inside each of the m connected components of $\{G > 0\}$. The same conclusion holds when $F(0, 0) < 0$ except that then the nodal set of the perturbation $G - \epsilon F$ lies in $\{G < 0\}$.

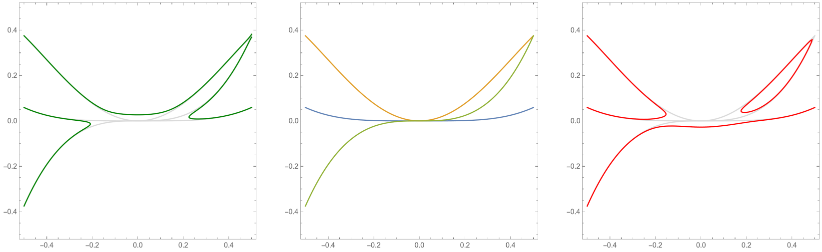


Figure: Zero set of $G(x, y) = (x^4 - y - y^2)(x^2(x^2 - 1) + \frac{1}{2}y)(3x^3 - y)$ and its perturbation $G - \epsilon F$: $\epsilon = 10^{-5}$, $F(x, y) = 1$ (left), $F(x, y) = -1$ (right).

The Case $d \geq 3$ is Odd

Let $p_d(x,t)$ denote the basic HCP in $\mathbb{R}^{1+1}$.

Theorem (B-Jeznach 2024)

Assume $d \geq 3$ is odd. For all sufficiently small $\epsilon > 0$ and $\alpha > 0$,

$$u_{\epsilon,\alpha}(x,y,t) := yp_{d-1}(x,t) - \epsilon p_d(x\cos\alpha - y\sin\alpha, t)$$

is a time-dependent hcp in $\mathbb{R}^{2+1}$ of degree d and $\mathcal{N} u_{\epsilon,\alpha} = 2$

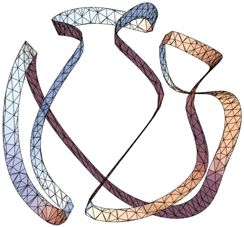


Figure: Nodal set when $d = 5$, $\epsilon = 0.3$, $\alpha = \pi/10$

Rewrite $u_{\epsilon,\alpha}|_{\mathbb{S}^2}(x, y, t)$ in spherical coordinates

Fix $\varepsilon > 0$ and $\alpha > 0$ (small) and write $u = u_{\epsilon,\alpha}$, $p = p_{d-1}$, $q = p_d$, and $q_\alpha(x, y, t) = p_d(x \cos\alpha - y \sin\alpha, t)$.

Consider the standard spherical coordinates on $\mathbb{S}^2$ given by

$$x = \cos\theta \cos\phi, \;\; y = \sin\theta \cos\phi, \;\; t = \sin\phi, \qquad -\pi < \theta \leq \pi, \;\; -\pi/2 \leq \phi \leq \pi/2$$

and write $\overline{p}$, $\overline{q}$, $\overline{q}_\alpha$, and $\overline{u}$ for the functions corresponding to $yp_d(x, t)$, $q(x, t)$, $q_\alpha(x, y, t)$, and $u_{\epsilon,\alpha}(x, y, t)$ on $\mathbb{S}^2$ written in spherical coordinates. Hence

$$\overline{p}(\theta, \phi) = \sin\theta \cos\phi \prod_{i=1}^{k} \left(\sin\phi + b_i \cos^2\theta \cos^2\phi\right),$$

$$\overline{q}(\theta, \phi) = \cos\theta \cos\phi \prod_{i=1}^{k} \left(\sin\phi + c_i \cos^2\theta \cos^2\phi\right),$$

$$\overline{q}_\alpha(\theta, \phi) = \overline{q}(\theta + \alpha, \phi), \quad \overline{u}(\theta, \phi) = \overline{p}(\theta, \phi) - \epsilon \overline{q}_\alpha(\theta, \phi).$$

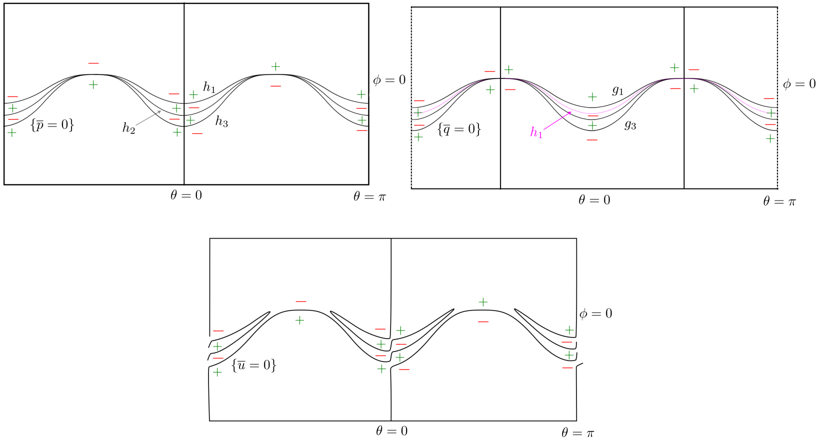


Figure: Proof of Theorem (1/2): Nodal set of $\overline{p}$ (top/left), $\overline{q}$ (top/right), and $\overline{u}$ (bottom) when $k = 3$ and ϵ and α are sufficiently small.

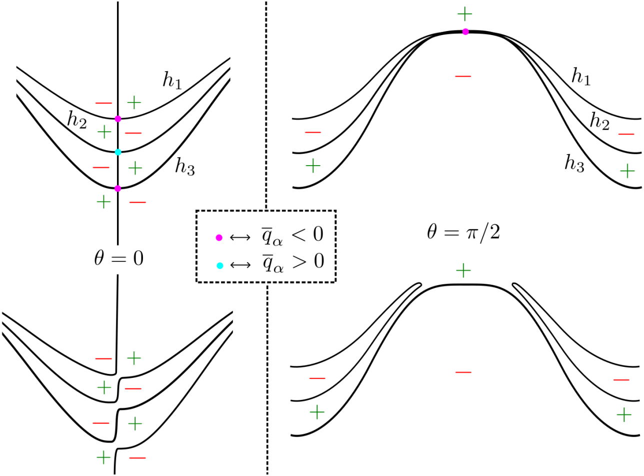


Figure: Proof of Theorem (2/2): Nodal sets of $\overline{p}$ (top) and $\overline{u}$ (bottom) near $\theta = 0$ (left) and $\theta = \pi/2$ (right) when $k = 3$. Sign of $\overline{q}_\alpha$ at singular points in nodal set of $\overline{p}$ determines local configuration of nodal domains of $\overline{u}$.

The Other Cases

Theorem (cf. Theorem 1 in Lewy (1977))

Assume $d = 4k + 2$ for some $k \geq 0$. Let $\psi(x, y) = \mathrm{Im}((x + iy)^d)$ and let $p_d(x, t)$ be the basic hcp in $\mathbb{R}^{1+1}$. For all sufficiently small $\epsilon > 0$,

$$u_\epsilon(x, y, t) := \psi(x, y) - \epsilon p_d(x, t)$$

is a time-dependent hcp in $\mathbb{R}^{2+1}$ of degree d and $\mathcal{N}(u_\epsilon) = 2$

Theorem (B-Jeznach 2024)

Assume $d = 4k$ for some $k \geq 1$. For small enough $\epsilon > 0$ and $\alpha > 0$,

$$\begin{aligned} u_{\epsilon,\alpha}(x, y, t) := & p_{2k}(x, t) p_{2k}(y, t) \\ & + \epsilon p_{2k+1}(x \cos\alpha - y \sin\alpha, t) p_{2k-1}(x \sin\alpha + y \cos\alpha, t) \end{aligned}$$

is a time-dependent hcp in $\mathbb{R}^{2+1}$ of degree d and $\mathcal{N}(u_{\epsilon,\alpha}) = 3$

Thank you for your attention!

Connecticut, Two Weeks Ago