Geometry of Measures and Anisotropic Square Functions

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Geometry of Measures

Goal: *Understand a measure on a space X through its interaction with families of distinguished subsets of X.*

Let X be a set and let $\mathcal{M}$ be a σ-algebra on X.
Let μ be a measure on $(X, \mathcal{M})$.
Let $\mathcal{N}$ be a nonempty collection of sets in $\mathcal{M}$.

- μ is carried by $\mathcal{N}$ if there exist countably many sets $\Gamma_i \in \mathcal{N}$ such that $\mu(\Gamma_i) > 0$ and $\mu\left(X \setminus \bigcup_i \Gamma_i\right) = 0$.
- μ is invisible to $\mathcal{N}$ if $\mu(\Gamma) = 0$ for every $\Gamma \in \mathcal{N}$.

Exercise (Decomposition Theorem)

If μ is σ-finite, then μ can be written uniquely as $\mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$ where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is invisible to $\mathcal{N}$.

- Proof of the Decomposition Theorem is abstract nonsense.
- **Main Problem**: Find measure-theoretic and/or geometric characterizations or constructions of $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{N}}^{\perp}$?

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Selected History

Besicovitch (1928)

$X = \mathbb{R}^2$, $\mu = \mathcal{H}^1 \llcorner E$, $\mathcal{N}$ = rectifiable curves in $\mathbb{R}^2$

- decomposition of 1-sets into **regular** and **irregular** sets
- measure-theoretic and geometric characterizations of regular and irregular 1-sets (density, tangents, projections)

Morse and Randolph (1944)

$X = \mathbb{R}^2$, $\mu \ll \mathcal{H}^1$, $\mathcal{N}$ = rectifiable curves in $\mathbb{R}^2$

- Complete analogues of Besicovitch's results for locally finite **absolutely continuous measures**

Federer (1947)

$X = \mathbb{R}^n$, $\mu \ll \mathcal{H}^m$ $(1 \leq m \leq n-1)$,
$\mathcal{N}$ = images of Lipschitz maps from $[0,1]^m$ into $\mathbb{R}^n$

- Many results — **Besicovitch-Federer Projection Theorem**
- Open: Does $0 < \lim_{r\to 0} \mathcal{H}^m(E \cap B(x,r))/r^m < \infty$ at $\mathcal{H}^m$-a.e. $x \in E$ imply that E countably $(\mathcal{H}^m, m)$ rectifiable?

Selected History

Mattila (1975), Preiss (1987)
$X = \mathbb{R}^n$, $\mu \ll \mathcal{H}^m$ $(1 \leq m \leq n-1)$,
$\mathcal{N}$ = images of Lipschitz maps from $[0,1]^m$ into $\mathbb{R}^n$

- Mattila proved the density conjecture for $\mu = \mathcal{H}^m \llcorner E$
- Preiss proved the density conjecture for $\mu \ll \mathcal{H}^m$ and introduced important new tools (tangent measures, ...)

David and Semmes (1991) $X = \mathbb{R}^n$, $\mu \ll \mathcal{H}^m$ $(1 \leq m \leq n-1)$,
$\mathcal{N}$ = images of bi-Lipschitz maps from subsets of $\mathbb{R}^m$ into $\mathbb{R}^n$

- Quantitative rectifiability — **uniformly rectifiable sets**
- Established strong connections between rectifiability and boundedness of singular integral operators

Azzam and Tolsa (2015)
$X = \mathbb{R}^n$, $\mu \ll \mathcal{H}^m$ $(1 \leq m \leq n-1)$, $\mathcal{N}$ = bi-Lipschitz images

- New square function characterization of ***m*-rectifiable sets** and ***m*-rectifiable absolutely continuous measures**

Rectifiable and Purely Unrectifiable Measures

Let μ be a Borel measure on $\mathbb{R}^n$ and let $m \geq 0$ be an integer. We say that μ is *m*-rectifiable if there exist countably many

- Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$ $\qquad [0,1]^0 = \{0\}$

such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_i f_i([0,1]^m)\right) = 0.$$

(Federer's terminology: $\mathbb{R}^n$ is countably (μ, m)-rectifiable.)

We say that μ is purely *m*-unrectifiable provided $\mu(f([0,1]^m)) = 0$ for every Lipschitz map $f : [0,1]^m \to \mathbb{R}^n$

- Every measure μ on $\mathbb{R}^n$ is *m*-rectifiable for all $m \geq n$
- A measure μ is 0-rectifiable iff $\mu = \sum_{i=1}^{\infty} c_i \delta_{x_i}$
- A measure μ is purely 0-unrectifiable iff μ is atomless.

Examples of Rectifiable Measures

- **Subsets of Lipschitz Images:** Let $f : [0,1]^m \to \mathbb{R}^n$ be Lipschitz. Then $\mathcal{H}^m \llcorner E$ is m-rectifiable for all $E \subseteq f([0,1])^m$.

- **Weighted Sums:** Suppose that $\mathcal{H}^m \llcorner E_i$ is m-rectifiable and $m_i \geq 0$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} m_i \mathcal{H}^m \llcorner E_i$ is m-rectifiable.

- **A Locally Infinite Rectifiable Measure:** Let $\ell_i \subset \mathbb{R}^2$ be the line through the origin meeting the x-axis at angle $\theta_i \in [0,\pi)$. Assume that $\#\{\theta_i : i \geq 1\} = \infty$. Then $\phi = \mathcal{H}^1 \llcorner \bigcup_{i=1}^{\infty} \ell_i$ is 1-rectifiable and σ-finite, but $\phi(B(0,r)) = \infty$ for all $r > 0$.

- **A Radon Example with Locally Infinite Support:** $\psi = \sum_{i=1}^{\infty} 2^{-i} \mathcal{H}^1 \llcorner \ell_i$ is a 1-rectifiable Radon measure, but $\mathcal{H}^1 \llcorner \operatorname{spt} \psi = \phi$ is locally infinite on neighborhoods of 0.

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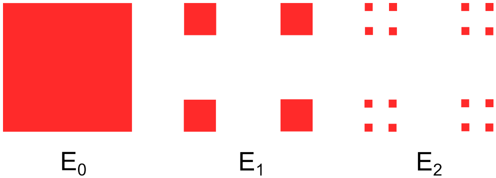
Examples of Purely Unrectifiable Measures

- Lebesgue measure on $\mathbb{R}^n$ is purely m-unrectifiable for all $m < n$ (This is obvious!)
- Let $E \subseteq \mathbb{R}^2$ be the "4 corners" Cantor set, $E = \bigcap_{i=0}^{\infty} E_i$



- Every rectifiable curve $\Gamma = f([0,1]) \subset \mathbb{R}^2$ intersects E in a set of zero $\mathcal{H}^1$ measure.
- $\mathcal{H}^1 \llcorner E$ is a purely 1-unrectifiable measure on $\mathbb{R}^2$

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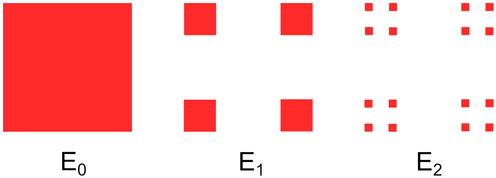

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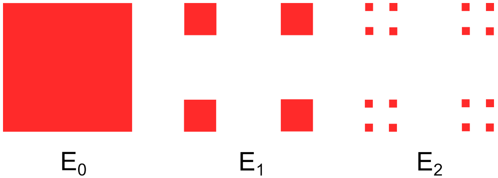

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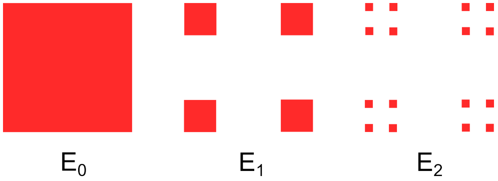

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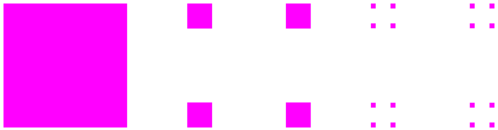
More Examples

Let $E_\lambda \subseteq \mathbb{R}^2$ be the generalized "4 corners" Cantor set, where $0 < \lambda \leq 1/2$ is the scaling factor.



$(\lambda = 1/5)$

- E has Hausdorff dimension $s = \log(4)/\log(1/\lambda)$
- $\mathcal{H}^s(E_\lambda \cap B(x,r)) \sim r^s$ for all $x \in E_\lambda$ and $0 < r < 1$.
- When $\lambda = 1/2$, $s = 2$ and $\mathcal{H}^s \llcorner E$ is just Lebesgue measure restricted to the unit square.
- If $1/4 \leq \lambda \leq 1/2$, then $\mathcal{H}^s \llcorner E_\lambda$ is purely 1-unrectifiable
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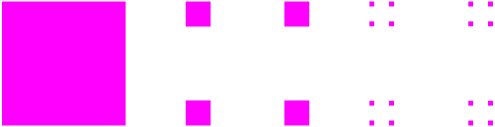

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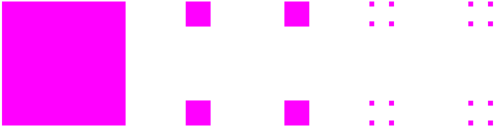

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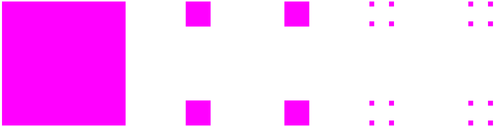

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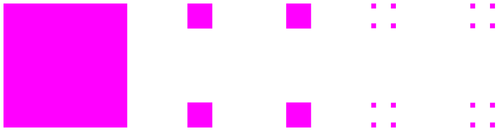

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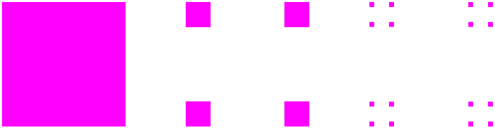

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Decomposition Theorem

Proposition Let μ be a Radon measure on $\mathbb{R}^n$. For each $m \geq 0$, we can write

$$\mu = \mu^m_{rect} + \mu^m_{pu},$$

where μ^m_{rect} is m-rectifiable and μ^m_{pu} is purely m-unrectifiable.

- $\mu^m_{rect} \perp \mu^m_{pu}$ and the decomposition is unique for each $m \geq 0$
- $\mu^m_{rect} = \mu$ and $\mu^m_{pu} = 0$ when $m \geq n$
- μ^0_{rect} is the atomic part of μ and μ^0_{pu} is the atomless part of μ
- The proof of this fact does not give a method to identify μ^m_{rect} and μ^m_{pu} when $1 \leq m \leq n-1$.

Problem Let $1 \leq m \leq n-1$. Give geometric, measure-theoretic characterizations of the m-rectifiable part μ^m_{rect} and the purely m-unrectifiable part μ^m_{pu} of Radon measures μ on $\mathbb{R}^n$.

Decomposition Theorem

Proposition Let μ be a Radon measure on $\mathbb{R}^n$. For each $m \geq 0$, we can write

$$\mu = \mu^m_{rect} + \mu^m_{pu},$$

where μ^m_{rect} is m-rectifiable and μ^m_{pu} is purely m-unrectifiable.

- $\mu^m_{rect} \perp \mu^m_{pu}$ and the decomposition is unique for each $m \geq 0$
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Grades of Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\cup\!\!\!\!\not\,$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

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$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E\ \}$$

"Absolutely continuous" rectifiable measures and rectifiable sets are very well understood through the work of Besicovitch, Morse and Randolph, Federer, Mattila, Preiss

In the absence of an absolute continuity assumption, rectifiable measures are poorly understood
(but there has been some significant progress when $m = 1$)

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Absolutely Continuous Rectifiable Measures ($1 \le m \le n-1$)

The lower and upper (Hausdorff) m-density of a measure μ at x:

$$\underline{D}^m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m} \quad \overline{D}^m(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m}.$$

Write $D^m(\mu, x)$, the m-density of μ at x, if $\underline{D}^m(\mu, x) = \overline{D}^m(\mu, x)$.

Theorem (Besicovitch 1928, Marstrand 1961, Mattila 1975)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m \llcorner E$ is locally finite. Then μ is m-rectifiable if and only if $D^m(\mu, x) = 1$ μ-a.e.

Theorem (Morse & Randolph 1944, Moore 1950, Preiss 1987)

Suppose μ is a locally finite Borel measure on $\mathbb{R}^n$ and $\mu \ll \mathcal{H}^m$. Then μ is m-rectifiable if and only if $0 < D^m(\mu, x) < \infty$ μ-a.e.

There are many other characterizations, see e.g. Federer (1947), Preiss (1987), Tolsa-Toro (2014), Tolsa & Azzam-Tolsa (2015)

Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ *on* $\mathbb{R}^n$ *(*$n \geq 2$*) with support* $\mathbb{R}^n$ *such that* $\mu \perp \mathcal{H}^1$*, but* μ *is 1-rectifiable.*

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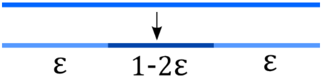

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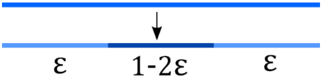

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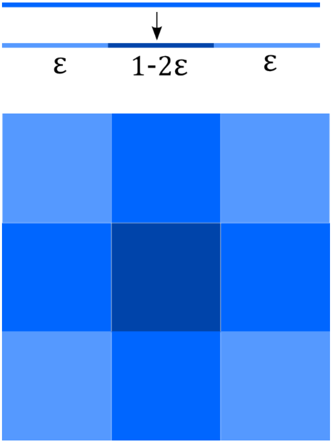

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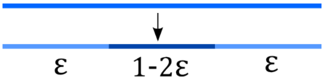

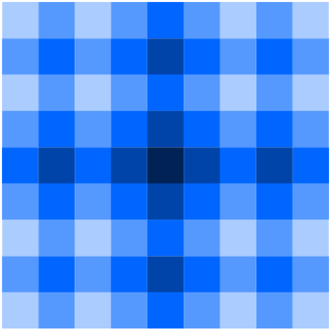

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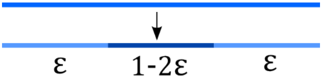

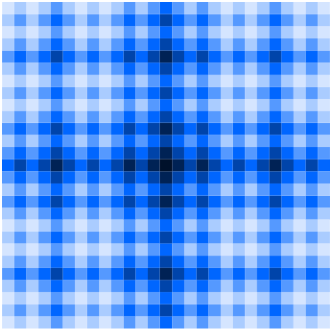

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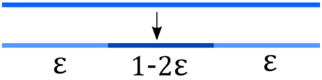

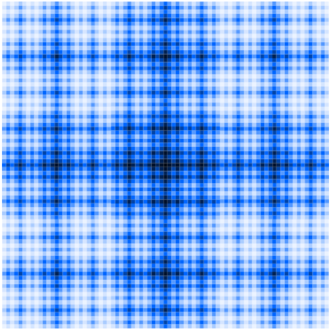

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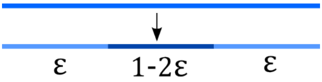

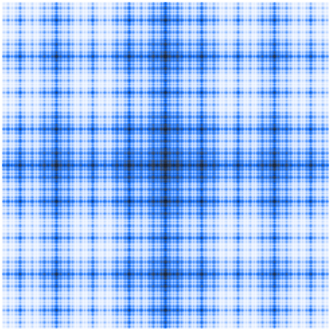

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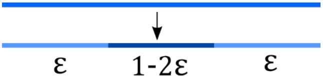

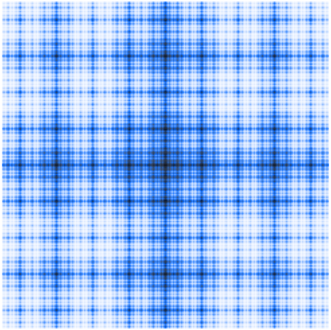

- $D^1(\mu, x) = \infty$ μ-a.e.
- $\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

  (see B-Schul PAMS 2016)
- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^n$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^n$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

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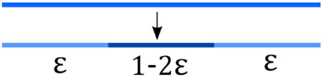

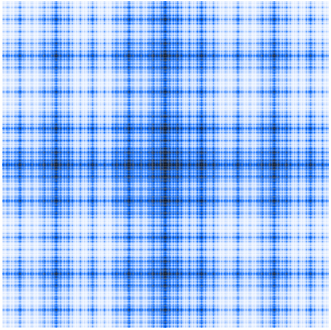

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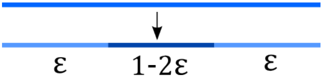

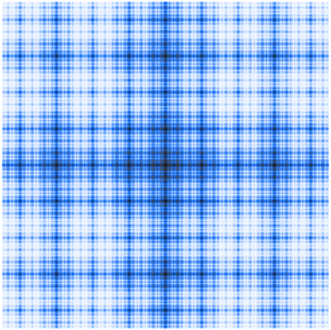

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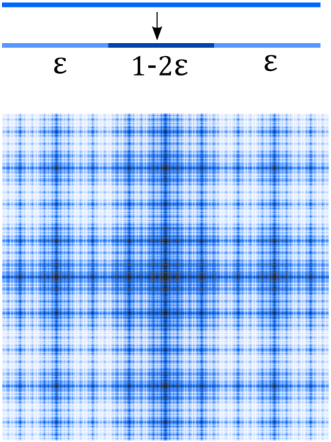

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General Rectifiable Measures

Partial Results: Necessary/Sufficient Conditions for $\mu = \mu^m_{rect}$

- Do not assume that $\mu \ll \mathcal{H}^m$

For 1-rectifiable measures

- Lerman (CPAM 2003) Sufficient conditions
- B and Schul (Math. Ann. 2015) Necessary conditions

For "badly linearly approximable" 1-rectifiable measures

- B and Schul (PAMS 2016) Characterization

For doubling 1-rectifiable measures with connected support

- Azzam and Mourgoglou (Anal. & PDE 2016) Characterization

For doubling *m*-rectifiable measures

- Azzam, David, Toro (Math. Ann. 2016) Sufficient conditions (a posteriori implies $\mu \ll \mathcal{H}^m$)

Old Results: Necessary Conditions

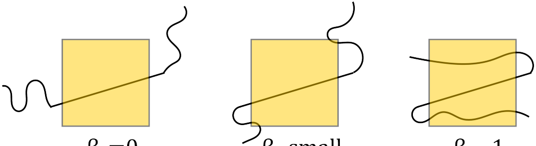
Theorem (B and Schul, Math. Ann. 2015)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \le p \le 2$.
If μ is 1-rectifiable, then $\underline{D}^1(\mu, x) > 0$ and $J_p(\mu, x) < \infty$ μ-a.e.

- $\underline{D}^1(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r}$ is lower 1-density of μ at x
- $J_p(\mu, x)$ is a geometric square function (or Jones function),

$$J_p(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \beta_p(\mu, \lambda Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x),$$

where $\beta_p(\mu, \lambda Q) \in [0, 1]$ is a measurement of L^p approximability of μ by a tangent line in a dilate λQ of Q.



$\beta_2 = 0$ β_2 small $\beta_2 \sim 1$

Here $\Delta(\mathbb{R}^n)$ denotes a fixed grid of half-open dyadic cubes in $\mathbb{R}^n$.

Old Results: Sufficient Conditions

- If μ is 1-rectifiable and $1 \le p \le 2$, then $J_p(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \beta_p(\mu, \lambda Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty$ μ-a.e.

Theorem (Pajot 1997 + B and Schul, PAMS 2016)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \le p < \infty$. Assume that $0 < \underline{D}^1(\mu, x) \le \overline{D}^1(\mu, x) < \infty$ μ-a.e. and

$$\sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \beta_p(\mu, Q)^2 \chi_Q(x) < \infty \quad \mu\text{-a.e.}.$$

Then μ is 1-rectifiable.

Theorem (B and Schul, PAMS 2016)

Let μ be a Radon measure on $\mathbb{R}^n$. Assume that

$$\sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x) < \infty \ \mu\text{-a.e.}$$

Then μ is 1-rectifiable.

New Result: Pointwise Doubling Measures

Theorem (B and Schul, arXiv 2016)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \le p \le 2$. Assume that $\limsup_{r\downarrow 0} \mu(B(x,2r))/\mu(B(x,r)) < \infty$ μ-a.e. Then:

$$\mu^1_{rect} = \mu \llcorner \{x \in \mathbb{R}^n : J_p(\mu, x) < \infty\}$$

$$\mu^1_{pu} = \mu \llcorner \{x \in \mathbb{R}^n : J_p(\mu, x) = \infty\}$$

- Recall

$$J_p(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \beta_p(\mu, \lambda Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x),$$

where $\beta_p(\mu, \lambda Q) \in [0,1]$ is a measurement of L^p approximability of μ by a tangent line in a dilate λQ of Q.

Here $\Delta(\mathbb{R}^n)$ denotes a fixed grid of half-open dyadic cubes in $\mathbb{R}^n$.

New Result: Characterization of μ^1_{rect} and μ^1_{pu}

Theorem (B and Schul, arXiv 2016)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \leq p \leq 2$. Then:

$$\mu^1_{rect} = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) > 0 \text{ and } J^*_p(\mu, x) < \infty\}$$

$$\mu^1_{pu} = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) = 0 \text{ or } J^*_p(\mu, x) = \infty\}$$

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$$J^*_p(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \leq 1}} \beta^*_p(\mu, Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x),$$

where $\beta^*_p(\mu, Q) \in [0, 1]$ is an anisotropic measurement of L^p approximability of μ by a tangent line in cubes "nearby" Q.

Here $\Delta(\mathbb{R}^n)$ denotes a fixed grid of half-open dyadic cubes in $\mathbb{R}^n$.

New Result: Traveling Salesman Theorem for Measures

Our methods yield characterization of **rectifiability of a measure with respect to a single curve**

Theorem (B and Schul, arXiv 2016)

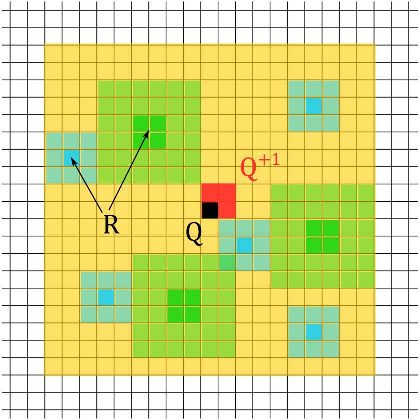
Let $1 \leq p < \infty$. *Let* μ *be a finite Borel measure on* $\mathbb{R}^n$ *with bounded support. Then there exists a rectifiable curve* $\Gamma = f([0,1])$, $f : [0,1] \to \mathbb{R}^n$ *Lipschitz, such that* $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ *if and only if*

$$S_p^{**}(\mu) = \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_p^{**}(\mu, Q)^2 \operatorname{diam} Q < \infty.$$

Moreover, the length of the shortest curve is comparable to $\operatorname{diam} \operatorname{spt} \mu + S_p^{**}(\mu)$ *up to constants depending only on* n.

- $\beta_p^{**}(\mu, Q)$ is a variant of $\beta_p^{*}(\mu, Q)$ — details soon
- Proof builds on the proof of the Traveling Salesman Theorem for Sets by Jones ($n = 2$) and Okikiolu ($n \geq 3$).

Nearby Cubes and $\beta_p^*(\mu, Q)$



For every dyadic cube $Q \subseteq \mathbb{R}^n$, the set $\Delta^*(Q)$ of nearby cubes are dyadic cubes R such that

- $3R \subseteq 1600\sqrt{n}Q$
- $\text{side}\, Q \le \text{side}\, R \le 2\,\text{side}\, Q$

Black cube represents cube Q and

Red cube Q^{+1} represents its parent

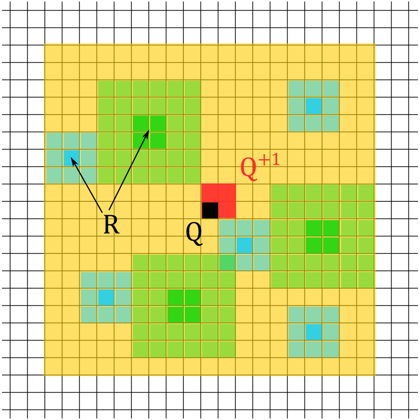
Yellow cube represents $1600\sqrt{n}Q$ (not to scale)

Cyan and green cubes represent cubes $R \in \Delta^*(Q)$ and their triples $3R$

Let μ be a Radon measure on $\mathbb{R}^n$, let $Q \subseteq \mathbb{R}^n$ be a dyadic cube, and let $1 \le p < \infty$. The beta number $\beta_p^*(\mu, Q) \in [0,1]$ is given by

$$\beta_p^*(\mu, Q)^p := \inf_{\text{lines } \ell} \sup_{R \in \Delta^*(Q)} \int_{3R} \left(\frac{\text{dist}(x,\ell)}{\text{diam}\, 3R} \right)^p \min\left(\frac{\mu(3R)}{\text{diam}\, 3R}, 1 \right)^{p/2} \frac{d\mu(x)}{\mu(3R)}$$

Nearby Cubes and $\beta_p^{**}(\mu, Q)$



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Anisotropic Measurements are Necessary for Non-Doubling Measures

It is natural to ask whether the anisotropic beta numbers $\beta^*_p(\mu, Q)$ (or $\beta^{**}_p(\mu, Q)$) are really necessary to characterize μ^1_{rect} and μ^1_{pu} for arbitrary Radon measures.

Could they be replaced by $\beta_p(\mu, \lambda Q)$? There is strong evidence that the answer is **No!**

Theorem (Martikainen and Orponen, arXiv 2016)

For all $\varepsilon > 0$, there exists a (non-doubling) measure μ on $\mathbb{R}^2$ s.t.

- $\operatorname{spt} \mu \subseteq [0,1]^2$ *and* $\mu(\mathbb{R}^2) = 1$
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There exists a finite, purely 1-unrectifiable measure μ on $\mathbb{R}^2$ with bounded support such that $\sum_{Q \in \Delta(\mathbb{R}^2)} \beta_2(\mu, \lambda Q)^2 \operatorname{diam} Q < \infty$.

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Proof Ingredient: Drawing Rectifiable Curves

Theorem (B and Schul, arXiv 2016)

Let $n \geq 2$, let $C^\star > 1$, let $x_0 \in \mathbb{R}^n$, and let $r_0 > 0$. Let $(V_k)_{k=0}^{\infty}$ be a sequence of nonempty finite subsets of $B(x_0, C^\star r_0)$ such that

1. *distinct points $v, v' \in V_k$ are uniformly separated: $|v - v'| \geq 2^{-k} r_0$;*
2. *for all $v_k \in V_k$, there exists $v_{k+1} \in V_{k+1}$ such that $|v_{k+1} - v_k| < C^\star 2^{-k} r_0$; and,*
3. *for all $v_k \in V_k$ ($k \geq 1$), there exists $v_{k-1} \in V_{k-1}$ such that $|v_{k-1} - v_k| < C^\star 2^{-k} r_0$.*

Suppose that for all $k \geq 1$ and for all $v \in V_k$ we are given a straight line $\ell_{k,v}$ in $\mathbb{R}^n$ and a number $\alpha_{k,v} \geq 0$ such that

$$\sup_{x \in (V_{k-1} \cup V_k) \cap B(v, 65 C^\star 2^{-k} r_0)} \operatorname{dist}(x, \ell_{k,v}) \leq \alpha_{k,v} 2^{-k} r_0 \tag{1}$$

and

$$\sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^2 2^{-k} r_0 < \infty. \tag{2}$$

Then the sets V_k converge in the Hausdorff metric to a compact set $V \subseteq \overline{B(x_0, C^\star r_0)}$ and there exists a compact, connected set $\Gamma \subseteq \overline{B(x_0, C^\star r_0)}$ such that $\Gamma \supseteq V$ and

$$\mathcal{H}^1(\Gamma) \lesssim_{n, C^\star} r_0 + \sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^2 2^{-k} r_0. \tag{3}$$

- This is a flexible criterion for drawing a rectifiable curve through the leaves of a tree; extends P. Jones' Traveling Salesman construction (Inventiones 1990), which required $V_{k+1} \supseteq V_k$
- Our write-up separates relatively simple description of the curve from the intricate length estimates

Takeaways

1. **Main Problem in Geometry of Measures:**
   Let $(X, \mathcal{M})$ be a measure space and let $\mathcal{N}$ be a family of measurable sets. Find geometric and/or measure-theoretic characterizations of measures that are
   - carried by $\mathcal{N}$ (rectifiable measures), or
   - invisible to $\mathcal{N}$ (purely unrectifiable measures).

   While this problem has been well-studied in $\mathbb{R}^n$ under certain regularity assumptions (absolutely continuous measures), there are many open questions when we drop regularity (Radon measures) or change the space X or sets $\mathcal{N}$.

2. **Anisotropic Measurements:**
   To hope to characterize geometric properties of non-doubling measures using multiscale quantities, it may be useful or even necessary to incorporate anisotropic normalizations.

Higher Dimensions?

Open Problem: *Find additional metric, geometric, and/or topological conditions which ensure that a compact, path-connected set $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^2(K) < \infty$ is contained in the image of a Lipschitz map $f : [0,1]^2 \to \mathbb{R}^n$.*

1. Reifenberg's algorithm
   - David and Toro, Reifenberg parameterizations for sets with holes, Memoirs of the Amer. Math. Soc., 2012.
   - Naber and Valtorta, Rectifiable-Reifenberg and the regularity of stationary and minimizing maps, arXiv 2015. To appear in Annals of Math.
   - Azzam and Schul, The Analyst's Traveling Salesman Theorem for sets of dimension larger than one, arXiv 2016.
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Current Project: Rectifiability in Non-integral Dimensions

Let $s \geq 1$ be a real number. A measure μ on $\mathbb{R}^n$ is **carried by (1/s)-Hölder continuous curves** if there exist countably many maps $f_i : [0,1] \to \mathbb{R}^n$ with $|f_i(x) - f_i(y)| \leq A_i|x-y|^{1/s}$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup f_i([0,1])\right) = 0.$$

Theorem (B 2016)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $s \geq 1$ be a real number. Then

$$\mu \llcorner \left\{ x \in \mathbb{R}^n : \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \leq 1}} \frac{(\text{diam } Q)^s}{\mu(Q)} \chi_Q(x) < \infty \right\}$$

is carried by $(1/s)$-Hölder continuous curves.

- The case $s = 1$ is due to B and Schul (PAMS 2016).
- When $s > 1$, it is necessary to construct Hölder continuous parameterization of curves capturing measure "by hand".

Thank you for listening!