Lipschitz Approximation to Corkscrew Domains

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Analysis on Rough Domains

Lipschitz Domains - $\Omega \subset \mathbb{R}^n$ locally area above a Lipschitz graph

1. Hunt and Wheeden (1960)
   Harmonic functions have non-tangential limits ω-a.e.
2. Dahlberg (1977)
   Harmonic measure ω is an A_∞ weight w.r.t. $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$.
   Hence $\omega(E) = 0 \Leftrightarrow \sigma(E) = 0$ for every Borel set $E \subset \partial\Omega$.
3. Coifman, McIntosh and Meyer (1982)
   Cauchy operator on a Lipschitz curve is bounded on L^2.

Rough Domains - do not admit parameterizations

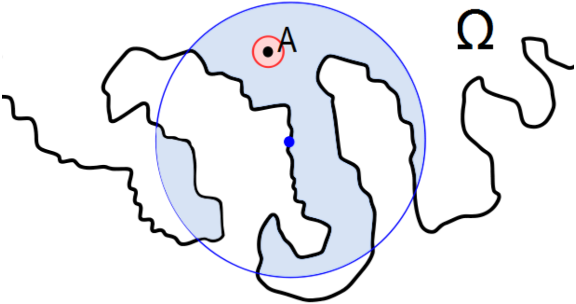
- Jerison and Kenig (1982)
  Boundary behavior of harmonic functions on NTA domains
- David and Jerison, Semmes (1990)
  (2), (3) on NTA domains + Ahlfors regular surface measure

The Corkscrew Condition

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then Ω has the **corkscrew condition** if there exists $M > 1$ and $R > 0$ such that:

- For every $Q \in \partial\Omega$, for every $0 < r < R$ there exists a **non-tangential point** $A \in \Omega \cap B(Q, r)$ such that

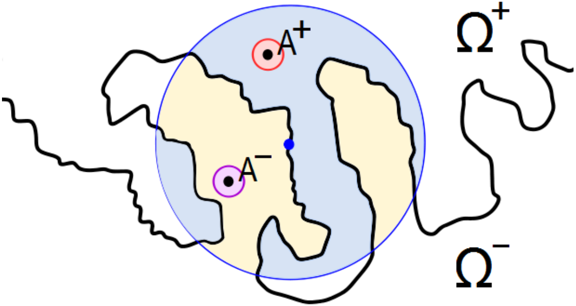
$$\mathrm{dist}(A, \partial\Omega) \geq r/M.$$



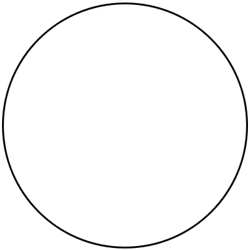
Corkscrew Domain

Let $\Omega \subset \mathbb{R}^n$ be an open set. Then Ω is a **corkscrew domain** if

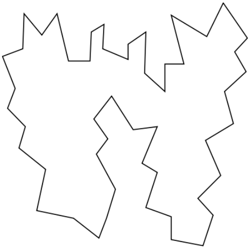
1. Ω is connected
2. $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ satisfy the corkscrew condition



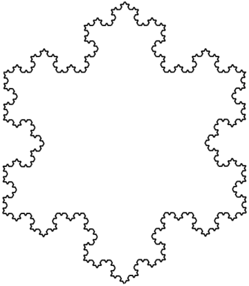
Examples of Corkscrew Domains



Smooth Domains



Lipschitz Domains

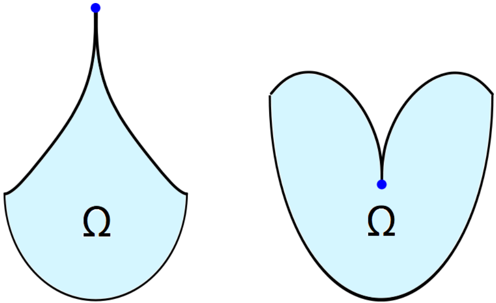


Quasispheres

(e.g. snowflake)

No Spikes in the Boundary

It is useful to see how the corkscrew condition may fail.



The domain on the left fails the interior corkscrew condition.
The domain on the right fails the exterior corkscrew condition.

Corkscrew Domain $\Rightarrow$ **No Spikes in Boundary**

Big Pieces of Lipschitz Graphs

A closed set $\Sigma \subset \mathbb{R}^n$ has **big pieces of Lipschitz graphs** if

1. Surface measure is Ahlfors regular on Σ:
   $C^{-1}r^{n-1} \leq \mathcal{H}^{n-1}(\Sigma \cap B(Q,r)) \leq Cr^{n-1}$ for all $0 < r < r_0$.
2. For every $Q \in \Sigma$ and $0 < r < r_0$ there is a Lipschitz graph Γ intersecting Σ in a big piece: $\mathcal{H}^{n-1}(\Gamma \cap \Sigma \cap B(Q,r)) \geq \varphi r^{n-1}$.

Theorem (David and Jerison)**.**
Suppose Σ satisfies (1) and $\mathbb{R}^n \setminus \Sigma$ satisfies a “two disk” condition.
Then Σ has big pieces of Lipschitz graphs.

Corollary. Let $\Sigma = \partial\Omega$ where $\Omega \subset \mathbb{R}^n$ is a corkscrew domain.
If surface measure $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$ is Ahlfors regular, then the boundary $\partial\Omega$ has big pieces of Lipschitz graphs.

Approximation Theorem

For corkscrew domains,

do not need Ahlfors regularity to build approximations!

Theorem (B)**.**
There is a constant $\psi = \psi(n, M) > 0$ with the following property.

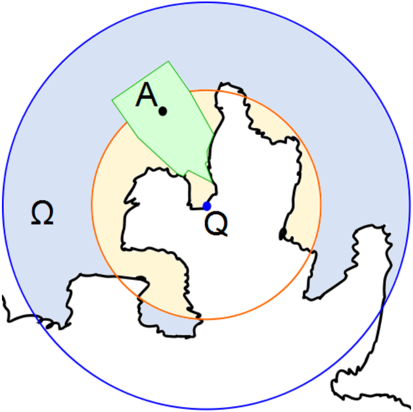
Let $\Omega \subset \mathbb{R}^n$ be a corkscrew domain with constants $M > 1$, $R > 0$.
For all $Q \in \partial\Omega$ and all $0 < r < R$ with $\mathcal{H}^{n-1}(\partial\Omega \cap B(Q, r)) < \infty$,

$(\star)$ for every non-tangential point $a = A^+(Q, r/2)$ there exists a Lipschitz domain $\Omega_L \subset \mathbb{R}^n$ such that $a \in \Omega_L \subset \Omega \cap B(Q, r)$ and $\mathcal{H}^{n-1}(\partial\Omega_L \cap \partial\Omega) \geq \psi r^{n-1}$.

Moreover,

- $\partial\Omega_L \cap \partial\Omega$ is contained in a Lipschitz graph,
- The Lipschitz constant and character of Ω_L depends only on n, M and $\gamma = \mathcal{H}^{n-1}(\partial\Omega \cap B(Q, r))/r^{n-1}$.

How do you construct good approximations?

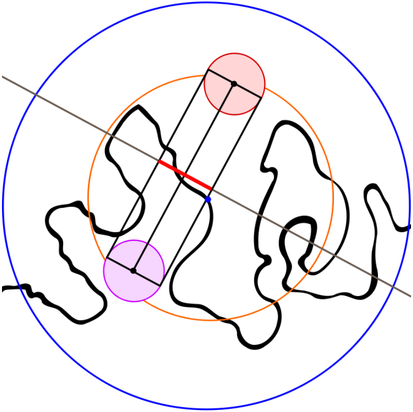


Ingredients in the Proof

1. Surface measure $\mathcal{H}^{n-1} \llcorner \partial\Omega$ on a corkscrew domain Ω is lower Ahlfors regular!
2. Basic Geometric Lemma: Use cones to identify sets contained in Lipschitz graphs.
3. The Maximal Theorem controls "jumps" in $\partial\Omega$. (This idea is due to David and Jerison.)

Surface Measure is Lower Ahlfors Regular

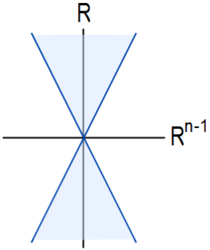
Lemma (B)**.** There exists $\beta = \beta(n, M) > 0$ such that:
for every corkscrew domain $\Omega \subset \mathbb{R}^n$ with constants $M > 1$, $R > 0$,
$\mathcal{H}^{n-1}(\partial\Omega \cap B(Q, r)) \geq \beta r^{n-1}$ for each $Q \in \partial\Omega$ and $0 < r < R$.

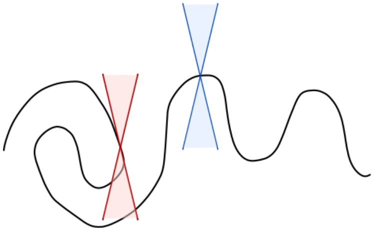


Identifying a Lipschitz Graph

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be projection onto first $(n-1)$-coordinates.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be projection onto the last coordinate.

Then $\mathcal{C} = \{y \in \mathbb{R}^n : |f(y)| \geq h|\pi(x)|\}$ is a cone with slope h.





Lemma. Let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then

$$\{y \in \Sigma : (y + \mathcal{C}) \cap \Sigma = \{y\}\}$$

sits in the graph of a function $F : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\text{Lip}(F) \leq h$.

A Variant with 1-sided Cones

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be projection onto first $(n-1)$-coordinates.
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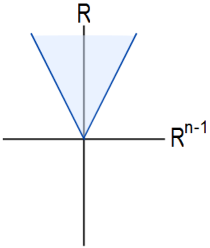

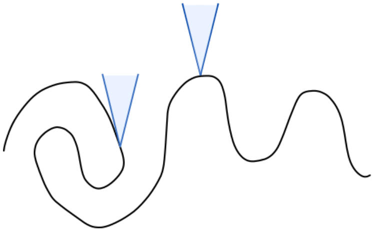

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The graph of F sits "above" Σ.

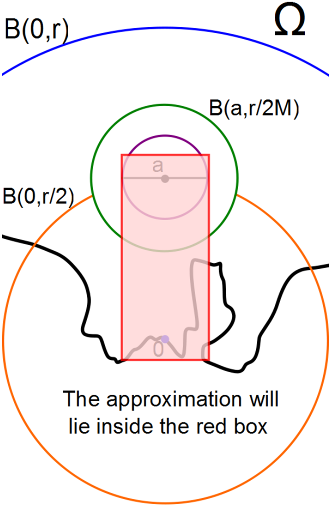
Building a Lipschitz Approximation to $\Omega \cap B(Q, r)$

Input:

- Corkscrew Domain Ω
- Location $Q \in \partial\Omega$
- Scale $0 < r < R$
- Upper Bound $\mathcal{H}^{n-1}(\partial\Omega \cap B(Q, r)) \leq \gamma r^{n-1}$
- Non-tangential Point $a = A^+(Q, r/2)$.

Output:

- Lipschitz Domain Ω_L
- $a \in \Omega_L \subset \Omega \cap B(Q, r))$
- Lower Bound $\mathcal{H}^{n-1}(\partial\Omega_L \cap \partial\Omega) \geq \psi r^{n-1}$



Building a Lipschitz Approximation to $\Omega \cap B(Q, r)$

Let $I_0 = [-s/2, s/2]^{n-1}$ is $(n-1)$-cube with side length $s = r/2M\sqrt{n-1}$.

Then $I_1 = I_0 \times \{a_n\} \subset B(a, r/4M)$.

Set T to be part of $\partial\Omega$ inside of the box $I_0 \times [-s/4, a_n - s/4]$.

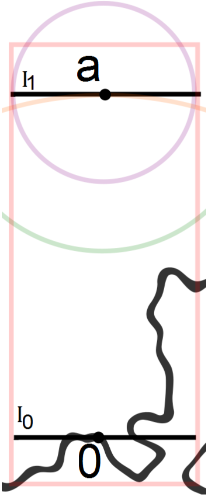
Fix a cone $\mathcal{C}$ opening upwards, $\mathcal{C} = \{y \in \mathbb{R}^n : f(y) > h|\pi(y)|\}$.

Idea. $T_\Gamma = \{y \in T : (y + \mathcal{C}) \cap \partial\Omega = \{y\}\}$ lies in Lipschitz graph $\Gamma = F(I_0)$ above T.

If $h = \mathrm{Lip}(F)$ large, then $\pi(T) \setminus \pi(T_\Gamma)$ small.

Lemma. $\mathcal{H}^{n-1}(\pi(T)) \geq c_{n,M} r^{n-1}$

Proposition. $\forall \varepsilon > 0\ \exists h = h(n, M, \gamma, \varepsilon) > 0$ such that $\pi(T) \setminus \pi(T_\Gamma) \leq \varepsilon r^{n-1}$.



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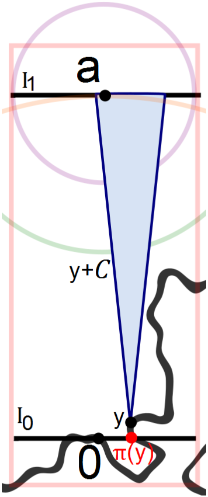

Building a Lipschitz Approximation to $\Omega \cap B(Q, r)$

Assign Ω_L to be the area above Γ and below l_1 plus the box $l_0 \times [a_n, a_n + s/4]$.

Set $\varepsilon = c_{n,M}/2$ and choose h accordingly.

$$\mathcal{H}^{n-1}(\partial\Omega_L \cap \partial\Omega) = \mathcal{H}^{n-1}(T_\Gamma) \geq \mathcal{H}^{n-1}(\pi(T_\Gamma)) \geq \mathcal{H}^{n-1}(\pi(T)) - \mathcal{H}^{n-1}(\pi(T) \backslash \pi(T_\Gamma)) \geq \frac{c_{n,M}}{2} r^{n-1}$$

Fix a cone $\mathcal{C}$ opening upwards, $\mathcal{C} = \{y \in \mathbb{R}^n : f(y) > h|\pi(y)|\}$.

Idea. $T_\Gamma = \{y \in T : (y + \mathcal{C}) \cap \partial\Omega = \{y\}\}$ lies in Lipschitz graph $\Gamma = F(l_0)$ above T.

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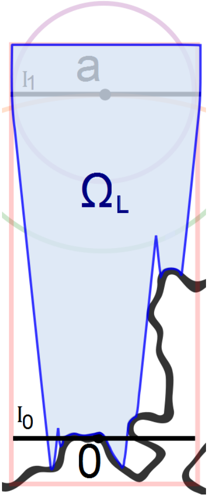
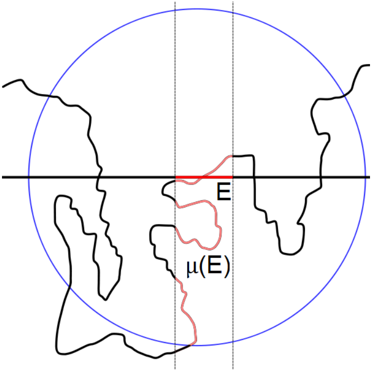


The Maximal Function



The **surface measure** of $\partial\Omega \cap B(Q, r)$ **over a set** $E \subset \mathbb{R}^{n-1}$ defines a finite measure μ on $\mathbb{R}^{n-1}$.

Define the maximal function $H : \mathbb{R}^{n-1} \to [0, \infty]$ by

$$H(x) = \sup_{I \ni x} \left\{ \frac{\mu(I)}{\mathcal{H}^{n-1}(I)} \right\}$$

where the sup is taken over all cubes $I \subset \mathbb{R}^{n-1}$ such that $x \in I$.

$$\lambda_H(N) = \{x \in \mathbb{R}^{n-1} : H(x) \geq N\}$$

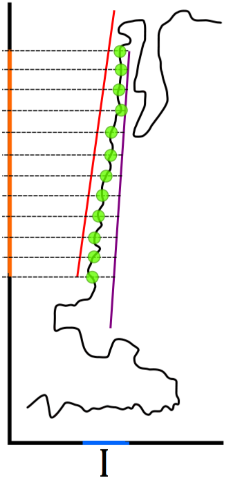
Maximal Theorem.

$$\mathcal{H}^{n-1}(\lambda_H(N)) \leq \frac{5^{n-1}\gamma r^{n-1}}{N}$$

(Choose N large, RHS $\leq \frac{\varepsilon}{2} r^{n-1}$.)

Big Jumps

If the surface $\partial\Omega \cap \pi^{-1}(I)$ over a cube $I \subset \mathbb{R}^{n-1}$ has a big vertical span relative to the width of I, the maximal function is big on I.



Lemma. Let $I \subset I_0$ cube of side length t. Suppose one can find line segments

- L in $B(Q,r) \cap \Omega$
- L' in $B(Q,r) \cap \Omega^c$

such that

- $\pi(L)$ and $\pi(L')$ belong to I
- $f(L) \cap f(L')$ is a segment of length $\geq \lceil 4^{n-1}\beta^{-1}N \rceil t$.

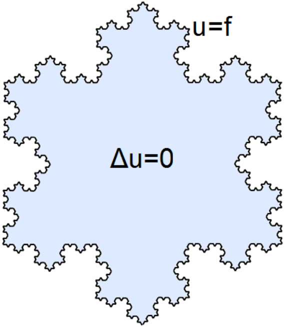
Then $H(x) \geq N$ for all $x \in I$.

The maximal theorem limits frequency of big vertical jumps in $\partial\Omega$ over I_0.

Harmonic Measure

Let $\Omega \subset \mathbb{R}^n$. Assume $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ satisfies corkscrew condition.

Then the Dirichlet problem



has a solution for every input $f \in C^2(\Omega) \cap C(\overline{\Omega})$.

Fix $X \in \partial\Omega$.
The maximum principle and Riesz representation theorem guarantees there exists a Borel probability measure ω^X such that

$$u_f(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q)$$

is the harmonic extension of f.

We call ω^X the **harmonic measure** of Ω with pole at X.

Note $\omega^X(E) = 0 \Leftrightarrow \omega^Y(E) = 0$ for all $X, Y \in \Omega$. (Harnack)

Null Sets of Harmonic Measure

Question. What conditions on the boundary guarantee harmonic measure ω and surface measure $\mathcal{H}^{n-1} \llcorner \partial\Omega$ have same null sets?

Theorem (F. and M. Riesz 1916)
Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain, bounded by a Jordan curve. If $\mathcal{H}^1(\partial\Omega) < \infty$, then

$$\omega(E) = 0 \Leftrightarrow \mathcal{H}^1(E) = 0 \quad \text{for every Borel set } E \subset \partial\Omega.$$

Example (Ziemer 1976)
There exists a topological sphere $\Omega \subset \mathbb{R}^3$ such that $\mathcal{H}^2(\partial\Omega) < \infty$ but ω is supported on a set of zero area.

Theorem (David-Jerison, Semmes 1990)
Let $\Omega \subset \mathbb{R}^n$ be NTA (corkscrew domain + Harnack chain condition)
If $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is Ahlfors regular, then $\omega \in A_\infty(\sigma)$.
In particular, $\omega(E) = 0 \Leftrightarrow \mathcal{H}^{n-1}(E) = 0$ for all $E \subset \partial\Omega$.

Null Sets of Harmonic Measure

The approximation theorem and Dahlberg's theorem $\Rightarrow$

Theorem (B).
Let $\Omega \subset \mathbb{R}^n$ be an NTA domain. Then the set

$$A = \left\{ Q \in \partial\Omega : \liminf_{r \downarrow 0} \frac{\mathcal{H}^{n-1}(\partial\Omega \cap B(Q,r))}{r^{n-1}} < \infty \right\}$$

has the property that $\omega(A \cap E) = 0 \Leftrightarrow \mathcal{H}^{n-1}(A \cap E) = 0$ for every Borel set $E \subset \partial\Omega$.

Corollary (B).
Let $\Omega \subset \mathbb{R}^n$ be an NTA domain. If $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then

$$\omega(E) = 0 \Rightarrow \mathcal{H}^{n-1}(E) = 0 \quad \text{for every Borel set } E \subset \partial\Omega.$$

Hausdorff Dimension of Harmonic Measure

Definition. The (upper) Hausdorff dimension of harmonic measure
$$\mathcal{H}\text{--}\dim \omega = \inf\{\dim_H E : \omega(E) = 1 \text{ and } E \subset \partial\Omega\}.$$

- Makarov (1985): If $\Omega \subset \mathbb{C}$ is simply connected, $\mathcal{H}\text{--}\dim \omega = 1$.
- For any domain $n - 2 \leq \mathcal{H}\text{--}\dim \omega \leq n - b_n$ for some $b_n > 0$. The upper bound is due to Bourgain (1987).
- Wolff (1995): There exist NTA domains $\Omega \subset \mathbb{R}^3$ such that $\mathcal{H}\text{--}\dim \omega > 2$ and other domains such that $\mathcal{H}\text{--}\dim \omega < 2$. These are now called **Wolff snowflakes**.
- Kenig, Preiss, Toro (2009): If $\Omega^+ = \Omega$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ are both NTA and $\omega^+ \ll \omega^- \ll \omega^+$, then $\mathcal{H}\text{--}\dim \omega = n - 1$.

Theorem (B)**.** Let $\Omega \subset \mathbb{R}^n$ be NTA.

1. If $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then $\mathcal{H}\text{--}\dim \omega = n - 1$.
2. If $\mathcal{H}\text{--}\dim \omega < n - 1$, then $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is locally infinite.

Open Problems

Corkscrew Domains

1. If we also assume $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is upper Ahlfors regular, can we improve the Lipschitz constant in the approximation theorem?
2. What is the correct dependence of $\beta(n, M)$ on M?

Harmonic Measure

3. Does F. and M. Riesz have a full analogue for NTA domains?

   Conjecture. Let $\Omega \subset \mathbb{R}^n$ be NTA. Assume $\mathcal{H}^{n-1}(\partial\Omega) < \infty$. Then

$$B = \left\{ Q \in \partial\Omega : \lim_{r \downarrow 0} \frac{\mathcal{H}^{n-1}(\partial\Omega \cap B(Q, r))}{r^{n-1}} = \infty \right\}$$

   has harmonic measure zero.