Tangent Measures and Harmonic Polynomials

Matthew Badger

University of Washington

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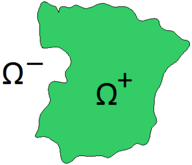
Harmonic Analysis, Geometric Measure Theory
and Quasiconformal Mappings

Centre de Recerca Matemàtica, Bellaterra, España

Harmonic Measure on 2-sided Domains

Let $\Omega \subset \mathbb{R}^n$ be 2-sided NTA domain
(e.g. quasiball)

- ω^+ harmonic measure of interior $\Omega^+ = \Omega$
- ω^- harmonic measure of exterior $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$



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- When are ω^+ and ω^- mutually absolutely continuous?
  (e.g. smooth domains, Lipschitz domains, chord arc domains)
- What conditions does $\omega^+ \ll \omega^- \ll \omega^+$ force on $\partial\Omega$?
- When is $\omega^+ \perp \omega^-$?
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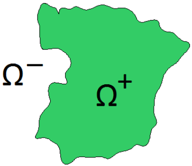

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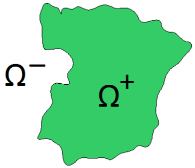

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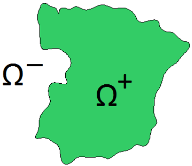

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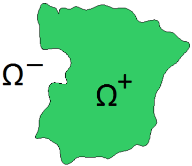

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Definition: The Hausdorff dimension of harmonic measure

$$\mathcal{H}\text{–}\dim \omega = \inf\{s : \exists E \subset \partial\Omega \text{ with } \mathcal{H}^s(E) = 0 \text{ and } \omega(\partial\Omega \setminus E) = 0\}$$

is the smallest dimension of a set with full harmonic measure

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Blow-ups of the Boundary are Homogeneous

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain.

Theorem (B)

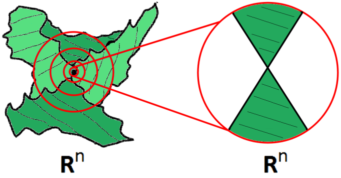
Assume $\omega^+ \ll \omega^- \ll \omega^+$ *and* $\log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega)$. *Then:*

$$\partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_d.$$

For each $Q \in \Gamma_k$ *and sequence* $r_i \downarrow 0$, *there exist a subsequence and a homogeneous harmonic polynomial* $h : \mathbb{R}^n \to \mathbb{R}$ *of degree* k *s.t.*

$$\frac{\partial\Omega - Q}{r_i} \to h^{-1}(0) \quad \text{(in Hausdorff distance).}$$

Example Polynomials: Linear Polynomials, $X_1^2 + X_2^2 - X_3^2 - X_4^2$



Blow-ups of the Boundary $\longleftrightarrow$ Tangent Measures of ω

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain, let $Q \in \partial\Omega$ and let $r_i \downarrow 0$.

Theorem: (KT) There is subsequence of r_i (which we relabel) and an unbounded 2-sided NTA domain Ω_∞ such that

- Blow-ups of Boundary at Q Converge:

$$\partial\Omega_i = \frac{\partial\Omega - Q}{r_i} \to \partial\Omega_\infty \text{ in Hausdorff metric}$$

- Blow-ups of Harmonic Measure at Q Converge:

$$\omega_i^\pm(E) = \frac{\omega^\pm(Q + r_i E)}{\omega^\pm(B(Q, r_i))} \text{ satisfy } \omega_i^\pm \rightharpoonup \omega_\infty^\pm$$

where $\omega_\infty^\pm$ is the harmonic measure of $\Omega_\infty^\pm$ with pole at infinity

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$$\partial\Omega_\infty = h^{-1}(0) \quad \text{and} \quad \Omega_\infty^\pm = \{X : h^\pm(X) > 0\}.$$

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$h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, $\Delta h = 0$

$\Omega^+ = \{X : h(X) > 0\}$, $\Omega^- = \{X : h(X) < 0\}$
(i.e. $h^\pm$ is the Green function for $\Omega^\pm$)

The harmonic measure ω_h associated to h is the harmonic measure of $\Omega^\pm$ with pole at infinity; i.e., for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{h^{-1}(0)} \varphi \, d\omega_h = -\int_{\partial\Omega^\pm} \varphi \frac{\partial h^\pm}{\partial \nu} d\sigma = \int_{\Omega^\pm} h^\pm \Delta\varphi$$

Two Collections of Measures Associated to Polynomials

$\mathcal{P}_d = \{\omega_h : h \text{ harmonic polynomial of degree} \leq d\}$

$\mathcal{F}_k = \{\omega_h : h \text{ homogenous harmonic polynomial of degree} = k\}$

Harmonic Measure Associated to a Harmonic Polynomial

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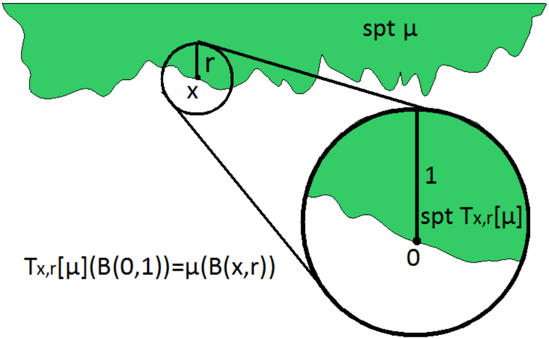
Translation and Dilation of Measures

- For $x \in \mathbb{R}^n$ and $r > 0$ define $T_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{x,r}(y) = \frac{y-x}{r}$.
- If μ Radon measure on $\mathbb{R}^n$ and $x \in \operatorname{spt} \mu$, the image measure $T_{x,r}[\mu]$ is defined by $T_{x,r}[\mu](E) = \mu(T^{-1}(E)) = \mu(x + rE)$.



Let μ Radon measure on $\mathbb{R}^n$, $x \in \operatorname{spt}\mu$.
We say ν is a tangent measure to μ at x, i.e., $\nu \in \operatorname{Tan}(\mu, x)$ if there exists sequences $r_i \downarrow 0$ and $c_i > 0$ such that

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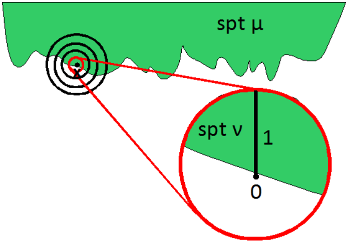

Example ($\Omega \subset \mathbb{R}^n$ 2-sided NTA domain)
– Let $Q \in \partial\Omega$ and $r_i \downarrow 0$. With the constants $c_i = \omega^+(B(Q, r_i))^{-1}$:

$$c_i T_{Q,r_i}[\omega^+] \rightharpoonup \omega^+_\infty \quad \text{(along subsequence).}$$

That is, $\omega^+_\infty \in \operatorname{Tan}(\omega^+, Q)$.

– If $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega)$, then there exists $d \geq 1$ such that $\operatorname{Tan}(\omega^\pm, Q) \subset \mathcal{P}_d$ for all $Q \in \partial\Omega$.

– **Goal:** $\operatorname{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ for some $1 \leq k(Q) \leq d$.

Tangent Measures

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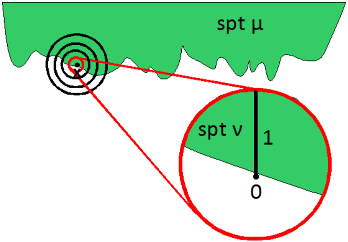

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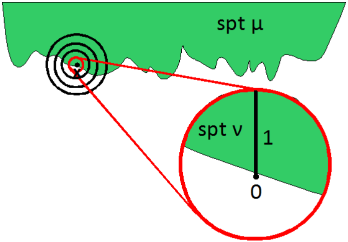

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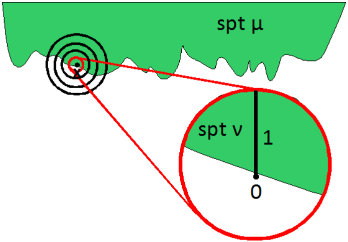

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A collection $\mathcal{M}$ of non-zero Radon measures is a d-cone if it preserved under scaling and dilation of $\mathbb{R}^n$:

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Examples

- Tangent Measures: $\mathrm{Tan}(\mu, x)$
- Polynomial Harmonic Measures: $\mathcal{P}_d$ and $\mathcal{F}_k$

Size of a Measure and Distance to a Cone

- Let ψ be Radon measure on $\mathbb{R}^n$. The "size" of ψ on $B(0,r)$ is

$$F_r(\psi) = \int_0^r \psi(B(0,s))ds.$$

- Let ψ be a Radon measure on $\mathbb{R}^n$ and $\mathcal{M}$ a d-cone. There is a "distance" $d_r(\psi, \mathcal{M})$ from ψ to $\mathcal{M}$ on $B(0,r)$ compatible with weak convergence of measures.

Cones of Measures

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Connectedness of Tangent Measures

Let $\mathcal{F}$ and $\mathcal{M}$ be d-cones such that $\mathcal{F} \subset \mathcal{M}$. Assume that:

- $\mathcal{F}$ and $\mathcal{M}$ have compact bases ($\{\psi : F_1(\psi) = 1\}$),
- (Property P) There exists $\epsilon_0 > 0$ such that whenever $\mu \in \mathcal{M}$ and $d_r(\mu, \mathcal{F}) < \epsilon_0$ for all $r \geq r_0$ then $\mu \in \mathcal{F}$.

Theorem

If $\mathrm{Tan}(\nu, x) \subset \mathcal{M}$ *and* $\mathrm{Tan}(\nu, x) \cap \mathcal{F} \neq \varnothing$, *then* $\mathrm{Tan}(\nu, x) \subset \mathcal{F}$.

Key Point: (Under technical hypotheses) If one tangent measure at a point belongs to $\mathcal{F}$ then all tangent measures belong to $\mathcal{F}$.

First proved in [P], the theorem was stated in this form in [KPT].

Checking the Hypotheses: Rate of Doubling

- If $\omega \in \mathcal{F}_k$, then $\omega(B(0,r)) = cr^{n+k-2}$ where c depends on n, k and $\|h\|_{L^1(S^{n-1})}$. Thus $\mathcal{F}_k$ is uniformly doubling: if $\omega \in \mathcal{F}_k$ then

$$\frac{\omega(B(0,2r))}{\omega(B(0,r))} = 2^{n+k-2} \quad \text{for all } r > 0$$

  independent of the associated polynomial h.
  Lemma: $\mathcal{F}_k$ has compact basis for all $k \geq 1$.

- If $\omega \in \mathcal{P}_d$ is associated to a polynomial of degree $j \leq d$ (not necessarily homogeneous), then for all $\tau > 1$

$$\frac{\omega(B(0,\tau r))}{\omega(B(0,r))} \sim \tau^{n+j-2} \quad \text{as } r \to \infty.$$

  Theorem: The comparison constant depends only on n and j!

  Corollary: If $d_r(\omega, \mathcal{F}_k) < \varepsilon_0(n,d)$ for all $r \geq r_0(\omega)$, then $k = j$.

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Polynomial Blow-ups are Homogeneous

Theorem (B)

Let Ω be a 2-sided NTA domain. If $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{P}_d$, *then* $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ *for some* $1 \le k \le d$.

Steps in the Proof

1. Since $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{P}_d$, there is a smallest degree $k \le d$ such that $\mathrm{Tan}(\omega^+, Q) \cap \mathcal{P}_k \ne \varnothing$. Show that $\mathrm{Tan}(\omega^+, Q) \cap \mathcal{P}_k \subset \mathcal{F}_k$.
2. Let $\mathcal{F} = \mathcal{F}_k$ and $\mathcal{M} = \mathrm{Tan}(\omega^+, Q) \cup \mathcal{F}_k$. By the previous slide the hypotheses of the connectedness theorem are satisfied. Therefore, $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{F}$.

Corollary: Let Ω be a 2-sided NTA domain. Assume $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in C(\partial\Omega)$. There exists $d \ge 1$ such that

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Open Questions

1. Assume $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$. Is Γ_1 open?
2. Does $\Gamma_1 = G \cup N$ where G is $(n-1)$-rectifiable and $\omega^{\pm}(N) = 0$?
3. It is not hard to show $\omega(\Gamma_2 \cup \cdots \cup \Gamma_d) = 0$.

   (The zero set of a harmonic polynomial is smooth except on a set of dimension $\leq n-2$; c.f. [HS]. Therefore, $\mathrm{Tan}(\omega, Q) \cap \mathcal{F}_1 \neq \varnothing$ at ω-a.e $Q \in \partial\Omega$, since "tangent measures to tangent measures are tangent measures.")

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Distance from Measure to a Cone

Let $\mathcal{L}(r) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \geq 0, \text{ Lip} f \leq 1, \text{ spt} f \subset B(0,r)\}$.

If μ and ν are two Radon measures in $\mathbb{R}^n$ and $r > 0$, we set

$$F_r(\mu,\nu) = \sup\left\{\left|\int f d\mu - \int f d\nu\right| : f \in \mathcal{L}(r)\right\}.$$

When $\nu = 0$,

$$F_r(\mu,0) = \int_0^r \mu(B(0,s))ds =: F_r(\mu).$$

Note that $\mu_i \rightharpoonup \mu$ if and only if $\lim_{i\to\infty} F_r(\mu_i,\mu) = 0$ for all $r > 0$.

If ψ is a Radon measure and $\mathcal{M}$ is a d-cone, we define a scaled version of F_r as follows:

$$d_r(\psi,\mathcal{M}) = \inf\left\{F_r\left(\frac{\psi}{F_r(\psi)},\mu\right) : \mu \in \mathcal{M} \text{ and } F_r(\mu) = 1\right\},$$

i.e., normalize ψ so $F_r(\psi) = 1$ & then take distance to $\mathcal{M}$ on B_r.

Homogeneous Harmonic Polynomials – "Big Piece" Lemma

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a homogenous harmonic polynomial of degree k.

Key Lemma: There is a constant $\ell_{n,k} > 0$ s.t. for all $t \in (0,1)$,

$$\mathcal{H}^{n-1}\{\theta \in S^{n-1} : |h(\theta)| \geq t\|h\|_{L^\infty(S^{n-1})}\} \geq \ell_{n,k}(1-t)^{n-1}.$$

Interpretation: If h is homogeneous harmonic polynomial,
then h takes big values on a big piece of the unit sphere.

Bounds for $\omega(B(0,r))$ as $r \to \infty$

Given $h : \mathbb{R}^n \to \mathbb{R}$ harmonic polynomial of degree d, $h(0) = 0$,

$$h = h_d + h_{d-1} + \cdots + h_1$$

where h_k is homogeneous harmonic polynomial of degree k.
In polar coordinates,

$$h(r\theta) = r^d h_d(\theta) + r^{d-1} h_{d-1}(\theta) + \cdots + rh_1(\theta),$$

$$\frac{dh}{dr}(r\theta) = dr^{d-1} h_d(\theta) + (d-1) r^{d-2} h_{d-1}(\theta) + \cdots + h_1(\theta).$$

Fact: Recall $\Omega^+ = \{X : h(X) > 0\}$. For all $r > 0$,

$$\omega(B(0,r)) = \int_{\partial B(0,r)\cap\Omega^+} \frac{dh^+}{dr} d\sigma$$

The $r^{d-1} h_d(\theta)$ term dominates as $r \to \infty$. Upper bounds for $\omega(B_r)$ are easy. Use "Big Piece" lemma to get uniform lower bounds.