Rectifiable and Purely Unrectifiable Measures in the Absence of Absolute Continuity

Matthew Badger
Joint work with Raanan Schul

Department of Mathematics
University of Connecticut

January 7, 2016

Joint Mathematics Meetings — Seattle, Washington

Special Session on Analysis and Geometry
in Nonsmooth Metric Measure Spaces

Research Partially Supported by NSF DMS 1500382

General Definition

Let μ be a Borel measure on $\mathbb{R}^n$ and let $m \geq 0$ be an integer. We say that μ is *m-rectifiable* if there exist countably many

- Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$ $\qquad [0,1]^0 = \{0\}$

such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_i f_i([0,1]^m)\right) = 0.$$

(Federer's terminology: $\mathbb{R}^n$ is countably (μ, m)-rectifiable.)

We say that μ is *purely m-unrectifiable* provided $\mu(f([0,1]^m)) = 0$ for every Lipschitz map $f : [0,1]^m \to \mathbb{R}^n$

- Every measure μ on $\mathbb{R}^n$ is m-rectifiable for all $m \geq n$
- A measure μ is 0-rectifiable iff $\mu = \sum_{i=1}^{\infty} c_i \delta_{x_i}$
- A measure μ is purely 0-unrectifiable iff μ is atomless.

Examples of Rectifiable Measures

- **Subsets of Lipschitz Images:** Let $f : [0,1]^m \to \mathbb{R}^n$ be Lipschitz. Then $\mathcal{H}^m \llcorner E$ is m-rectifiable for all $E \subseteq f([0,1])^m$.

- **Weighted Sums:** Suppose that $\mathcal{H}^m \llcorner E_i$ is m-rectifiable and $m_i \geq 0$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} m_i \mathcal{H}^m \llcorner E_i$ is m-rectifiable.

- **A Locally Infinite Rectifiable Measure:** Let $\ell_i \subset \mathbb{R}^2$ be the line through the origin meeting the x-axis at angle $\theta_i \in [0, \pi)$. Assume that $\#\{\theta_i : i \geq 1\} = \infty$. Then $\phi = \mathcal{H}^1 \llcorner \bigcup_{i=1}^{\infty} \ell_i$ is 1-rectifiable and σ-finite, but $\phi(B(0,r)) = \infty$ for all $r > 0$.

- **A Radon Example with Locally Infinite Support:** $\psi = \sum_{i=1}^{\infty} 2^{-i} \mathcal{H}^1 \llcorner \ell_i$ is a 1-rectifiable Radon measure, but $\mathcal{H}^1 \llcorner \operatorname{spt} \psi = \phi$ is locally infinite on neighborhoods of 0.

Examples of Rectifiable Measures

- **Subsets of Lipschitz Images:** Let $f : [0,1]^m \to \mathbb{R}^n$ be Lipschitz. Then $\mathcal{H}^m \llcorner E$ is m-rectifiable for all $E \subseteq f([0,1])^m$.

- **Weighted Sums:** Suppose that $\mathcal{H}^m \llcorner E_i$ is m-rectifiable and $m_i \geq 0$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} m_i \mathcal{H}^m \llcorner E_i$ is m-rectifiable.

- **A Locally Infinite Rectifiable Measure:** Let $\ell_i \subset \mathbb{R}^2$ be the line through the origin meeting the x-axis at angle $\theta_i \in [0,\pi)$. Assume that $\#\{\theta_i : i \geq 1\} = \infty$. Then $\phi = \mathcal{H}^1 \llcorner \bigcup_{i=1}^{\infty} \ell_i$ is 1-rectifiable and σ-finite, but $\phi(B(0,r)) = \infty$ for all $r > 0$.

- **A Radon Example with Locally Infinite Support:** $\psi = \sum_{i=1}^{\infty} 2^{-i} \mathcal{H}^1 \llcorner \ell_i$ is a 1-rectifiable Radon measure, but $\mathcal{H}^1 \llcorner \operatorname{spt} \psi = \phi$ is locally infinite on neighborhoods of 0.

Examples of Rectifiable Measures

- **Subsets of Lipschitz Images:** Let $f : [0,1]^m \to \mathbb{R}^n$ be Lipschitz. Then $\mathcal{H}^m \llcorner E$ is m-rectifiable for all $E \subseteq f([0,1])^m$.

- **Weighted Sums:** Suppose that $\mathcal{H}^m \llcorner E_i$ is m-rectifiable and $m_i \geq 0$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} m_i \mathcal{H}^m \llcorner E_i$ is m-rectifiable.

- **A Locally Infinite Rectifiable Measure:** Let $\ell_i \subset \mathbb{R}^2$ be the line through the origin meeting the x-axis at angle $\theta_i \in [0, \pi)$. Assume that $\#\{\theta_i : i \geq 1\} = \infty$. Then $\phi = \mathcal{H}^1 \llcorner \bigcup_{i=1}^{\infty} \ell_i$ is 1-rectifiable and σ-finite, but $\phi(B(0,r)) = \infty$ for all $r > 0$.

- **A Radon Example with Locally Infinite Support:** $\psi = \sum_{i=1}^{\infty} 2^{-i} \mathcal{H}^1 \llcorner \ell_i$ is a 1-rectifiable Radon measure, but $\mathcal{H}^1 \llcorner \operatorname{spt} \psi = \phi$ is locally infinite on neighborhoods of 0.

Examples of Rectifiable Measures

- **Subsets of Lipschitz Images:** Let $f : [0,1]^m \to \mathbb{R}^n$ be Lipschitz. Then $\mathcal{H}^m \llcorner E$ is m-rectifiable for all $E \subseteq f([0,1])^m$.

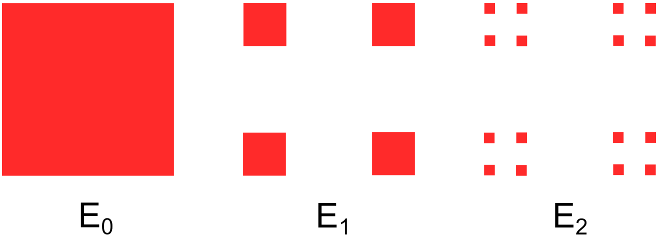
- **Weighted Sums:** Suppose that $\mathcal{H}^m \llcorner E_i$ is m-rectifiable and $m_i \geq 0$ for all $i \geq 1$. Then $\sum_{i=1}^{\infty} m_i \mathcal{H}^m \llcorner E_i$ is m-rectifiable.

- **A Locally Infinite Rectifiable Measure:** Let $\ell_i \subset \mathbb{R}^2$ be the line through the origin meeting the x-axis at angle $\theta_i \in [0, \pi)$. Assume that $\#\{\theta_i : i \geq 1\} = \infty$. Then $\phi = \mathcal{H}^1 \llcorner \bigcup_{i=1}^{\infty} \ell_i$ is 1-rectifiable and σ-finite, but $\phi(B(0,r)) = \infty$ for all $r > 0$.

- **A Radon Example with Locally Infinite Support:** $\psi = \sum_{i=1}^{\infty} 2^{-i} \mathcal{H}^1 \llcorner \ell_i$ is a 1-rectifiable Radon measure, but $\mathcal{H}^1 \llcorner \operatorname{spt} \psi = \phi$ is locally infinite on neighborhoods of 0.

Examples of Purely Unrectifiable Measures

Let $E \subseteq \mathbb{R}^2$ be the "4 corners" Cantor set, $E = \bigcap_{i=0}^{\infty} E_i$



- Every rectifiable curve $\Gamma = f([0,1]) \subset \mathbb{R}^2$ intersects E in a set of zero $\mathcal{H}^1$ measure.
- $\mathcal{H}^1 \llcorner E$ is a purely 1-unrectifiable measure on $\mathbb{R}^2$

  $\mathcal{H}^2 \llcorner (E \times \mathbb{R})$ is a purely 2-unrectifiable measure on $\mathbb{R}^3$

  and so on...

Decomposition Theorem

Proposition Let μ be a Radon measure on $\mathbb{R}^n$. For each $m \geq 0$, we can write

$$\mu = \mu^m_{rect} + \mu^m_{pu},$$

where μ^m_{rect} is m-rectifiable and μ^m_{pu} is purely m-unrectifiable.

- $\mu^m_{rect} \perp \mu^m_{pu}$ and the decomposition is unique for each $m \geq 0$
- $\mu^m_{rect} = \mu$ and $\mu^m_{pu} = 0$ when $m \geq n$
- μ^0_{rect} measures atoms of μ and μ^0_{pu} is the atomless part of μ
- The proof of this fact does not give a method to identify μ^m_{rect} and μ^m_{pu} when $1 \leq m \leq n-1$.

Problem Let $1 \leq m \leq n-1$. Give geometric, measure-theoretic characterizations of the m-rectifiable part μ^m_{rect} and the purely m-unrectifiable part μ^m_{pu} of Radon measures μ on $\mathbb{R}^n$.

Decomposition Theorem

Proposition Let μ be a Radon measure on $\mathbb{R}^n$. For each $m \geq 0$, we can write

$$\mu = \mu^m_{rect} + \mu^m_{pu},$$

where μ^m_{rect} is m-rectifiable and μ^m_{pu} is purely m-unrectifiable.

- $\mu^m_{rect} \perp \mu^m_{pu}$ and the decomposition is unique for each $m \geq 0$
- $\mu^m_{rect} = \mu$ and $\mu^m_{pu} = 0$ when $m \geq n$
- μ^0_{rect} measures atoms of μ and μ^0_{pu} is the atomless part of μ
- The proof of this fact does not give a method to identify μ^m_{rect} and μ^m_{pu} when $1 \leq m \leq n-1$.

Problem Let $1 \leq m \leq n-1$. Give geometric, measure-theoretic characterizations of the m-rectifiable part μ^m_{rect} and the purely m-unrectifiable part μ^m_{pu} of Radon measures μ on $\mathbb{R}^n$.

Grades of Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\cup\!\!\!\!\not\!\!$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

$$\cup\!\!\!\!\not\!\!$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E\ \}$$

"Absolutely continuous" rectifiable measures and rectifiable sets are very well understood

In the absence of an absolute continuity assumption, rectifiable measures are poorly understood

Grades of Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\supsetneq$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

$$\supsetneq$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E\ \}$$

"Absolutely continuous" rectifiable measures and rectifiable sets are very well understood

In the absence of an absolute continuity assumption, rectifiable measures are poorly understood

Grades of Rectifiable Measures

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n\ \}$$

$$\cup\!\!\!\!\!\not\;\;$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ such that } \mu \ll \mathcal{H}^m\ \}$$

$$\cup\!\!\!\!\!\not\;\;$$

$$\{\ m\text{-rectifiable measures } \mu \text{ on } \mathbb{R}^n \text{ of the form } \mu = \mathcal{H}^m \llcorner E\ \}$$

"Absolutely continuous" rectifiable measures and rectifiable sets are very well understood

In the absence of an absolute continuity assumption, rectifiable measures are poorly understood

Absolutely Continuous Rectifiable Measures ($1 \leq m \leq n-1$)

The lower and upper (Hausdorff) m-density of a measure μ at x:

$$\underline{D}^m(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m} \quad \overline{D}^m(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{c_m r^m}.$$

Write $D^m(\mu, x)$, the m-density of μ at x, if $\underline{D}^m(\mu, x) = \overline{D}^m(\mu, x)$.

Theorem (Besicovitch 1928, Marstrand 1961, Mattila 1975)

Suppose that $E \subset \mathbb{R}^n$ is Borel and $\mu = \mathcal{H}^m \llcorner E$ is locally finite. Then μ is m-rectifiable if and only if $D^m(\mu, x) = 1$ μ-a.e.

Theorem (Morse & Randolph 1944, Moore 1950, Preiss 1987)

Suppose μ is a locally finite Borel measure on $\mathbb{R}^n$ and $\mu \ll \mathcal{H}^m$. Then μ is m-rectifiable if and only if $0 < D^m(\mu, x) < \infty$ μ-a.e.

There are many other characterizations, see e.g. Federer (1947), Preiss (1987), Tolsa-Toro (2014), Tolsa & Azzam-Tolsa (2015)

Singular Rectifiable Measures

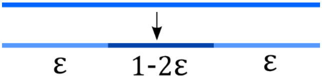
Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

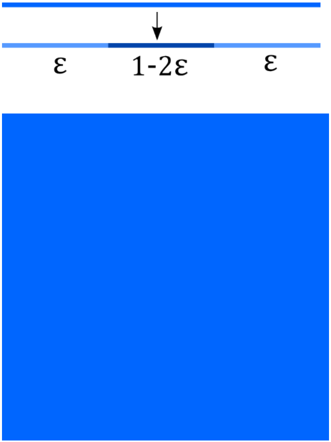
There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.



Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

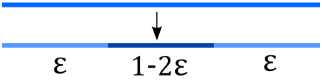
There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

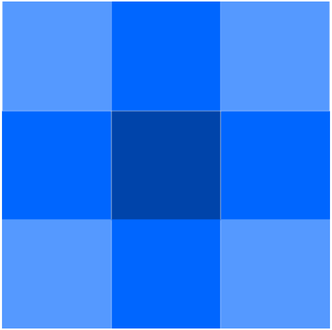


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

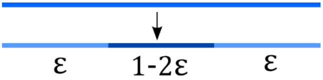


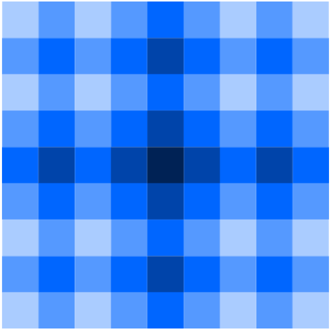


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

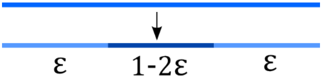


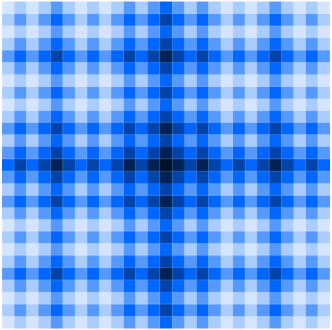


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

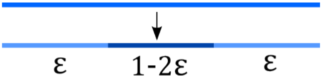


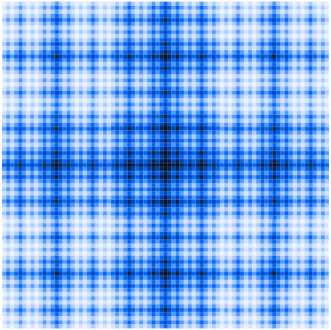


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

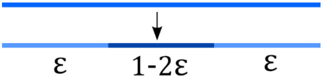


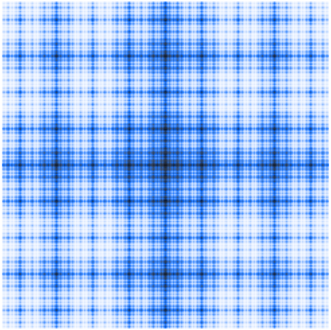


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.

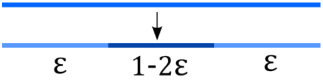


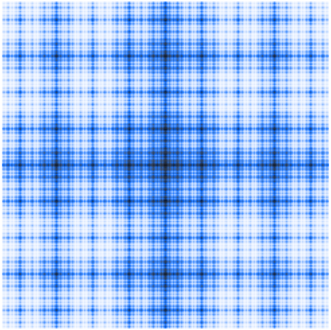


Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.





- $D^1(\mu, x) = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

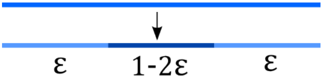
  (see B-Schul 2015)
- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^n$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^n$ such that

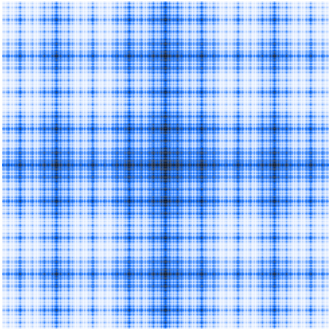
$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.





- $D^1(\mu, x) = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

(see B-Schul 2015)

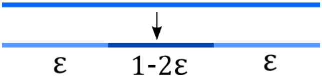
- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^n$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^n$ such that

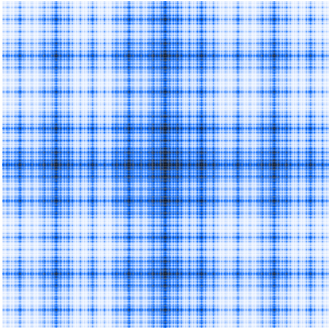
$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.





- $D^1(\mu, x) = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

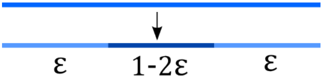
  (see B-Schul 2015)
- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^n$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^n$ such that

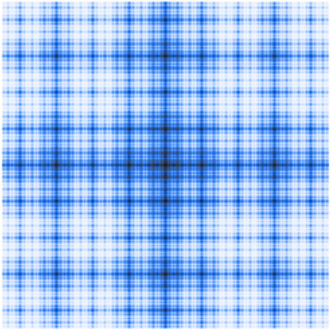
$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

Singular Rectifiable Measures

Theorem (Garnett-Killip-Schul 2010)

There exist a doubling measure μ on $\mathbb{R}^n$ ($n \geq 2$) with support $\mathbb{R}^n$ such that $\mu \perp \mathcal{H}^1$, but μ is 1-rectifiable.





- $D^1(\mu, x) = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

  (see B-Schul 2015)
- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^n$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^n$ such that

$$\mu\left(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

General Rectifiable Measures

Partial Results: Necessary/Sufficient Conditions for $\mu = \mu^m_{rect}$

- Do not assume that $\mu \ll \mathcal{H}^m$

For 1-rectifiable measures

- Lerman (2003) Sufficient conditions
- B and Schul (2015) Necessary conditions

For "badly linearly approximable" 1-rectifiable measures

- B and Schul (arXiv 2014) Characterization

For doubling 1-rectifiable measures with connected support

- Azzam and Mourgoglou (arXiv 2015) Characterization

For doubling *m*-rectifiable measures

- Azzam, David, Toro (arXiv 2014) Sufficient conditions
(a posteriori implies $\mu \ll \mathcal{H}^m$)

New Result: Characterization of μ^1_{rect} and μ^1_{pu}

Theorem (B and Schul)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \leq p \leq 2$. Then:

$$\mu^1_{rect} = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) > 0 \text{ and } J^*_p(\mu, x) < \infty\}$$

$$\mu^1_{pu} = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) = 0 \text{ or } J^*_p(\mu, x) = \infty\}$$

- $\underline{D}^1(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r}$ is lower 1-density of μ at x
- $J^*_p(\mu, x)$ is a geometric square function (or Jones function),

$$J^*_p(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \leq 1}} \beta^*_p(\mu, Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x),$$

where $\beta^*_p(\mu, Q) \in [0, 1]$ is a measurement of L^p approximability of μ by a tangent line in triples of dyadic cubes “nearby” Q.

Here $\Delta(\mathbb{R}^n)$ denotes a fixed grid of half-open dyadic cubes in $\mathbb{R}^n$.

New Result: Traveling Salesman Theorem for Measures

For lower ADR measures, our methods yield characterization of **rectifiability of a measure with respect to a single curve**

Theorem (B and Schul)

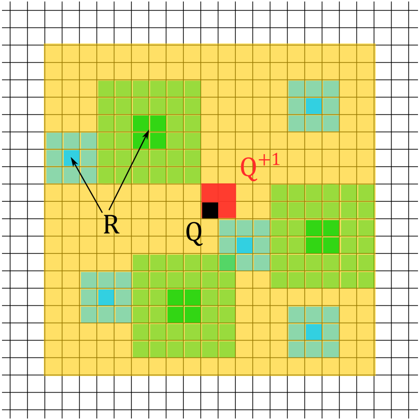
Let $1 \leq p \leq 2$. Let μ be a finite Borel measure on $\mathbb{R}^n$ with bounded support. Assume that $\mu(B(x,r)) \geq cr$ for all $x \in \operatorname{spt}\mu$ and for all $r \leq \operatorname{diam}\operatorname{spt}\mu$. Then there exists a rectifiable curve $\Gamma = f([0,1])$, $f : [0,1] \to \mathbb{R}^n$ Lipschitz, such that $\mu(\mathbb{R}^n \setminus \Gamma) = 0$ if and only if

$$S^2(\mu) = \sum_{Q \in \Delta(\mathbb{R}^n)} \beta_p^*(\mu, Q)^2 \operatorname{diam} Q < \infty.$$

Moreover, the length of the shortest curve is comparable to $\operatorname{diam}\operatorname{spt}\mu + S^2(\mu)$ up to constants depending only on n, c, and p.

- Proof builds on the proof of the Traveling Salesman Theorem for Sets by Jones ($n = 2$) and Okikiolu ($n \geq 3$).

Nearby Cubes and $\beta_p^*(\mu, Q)$



For every dyadic cube $Q \subseteq \mathbb{R}^n$, the set $\Delta^*(Q)$ of nearby cubes are dyadic cubes R such that

- $3R \subseteq 2000\sqrt{n}Q$
- $\text{side}\, Q \leq \text{side}\, R \leq 2\,\text{side}\, Q$

Black cube represents cube Q and

Red cube Q^{+1} represents its parent

Yellow cube represents $2000\sqrt{n}Q$ (not to scale)

Cyan and green cubes represent cubes $R \in \Delta^*(Q)$ and their triples $3R$

Let μ be a Radon measure on $\mathbb{R}^n$, let $Q \subseteq \mathbb{R}^n$ be a dyadic cube, and let $1 \leq p < \infty$. The beta number $\beta_p^*(\mu, Q) \in [0,1]$ is given by

$$\beta_p^*(\mu, Q)^p := \inf_{\text{lines } \ell} \sup_{R \in \Delta^*(Q)} \int_{3R} \left(\frac{\text{dist}(x,\ell)}{\text{diam}\, 3R}\right)^p \min\left(\frac{\mu(3R)}{\text{diam}\, 3R}, 1\right) \frac{d\mu(x)}{\mu(3R)}$$

Remarks

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \le p \le 2$

$$\Delta^*(Q) = \{\textit{dyadic cubes } R : 3R \subseteq 2000\sqrt{n}Q, \text{side } Q \le \text{side } R \le 2\,\text{side } Q\}$$

$$\beta_p^*(\mu, Q)^p := \inf_{\textit{lines } \ell} \sup_{R \in \Delta^*(Q)} \int_{3R} \left(\frac{\text{dist}(x,\ell)}{\text{diam } 3R}\right)^p \min\left(\frac{\mu(3R)}{\text{diam } 3R}, 1\right) \frac{d\mu(x)}{\mu(3R)}$$

$$\underline{D}^1(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r}, \qquad J_p^*(\mu, x) = \sum_{\substack{Q \in \Delta(\mathbb{R}^n) \\ \text{side } Q \le 1}} \beta_p^*(\mu, Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x)$$

$$\mu_{rect}^1 = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) > 0 \textit{ and } J_p^*(\mu, x) < \infty\}$$

$$\mu_{pu}^1 = \mu \llcorner \{x \in \mathbb{R}^n : \underline{D}^1(\mu, x) = 0 \textit{ or } J_p^*(\mu, x) = \infty\}$$

- "Sufficient condition for 1-rectifiable" uses new criterion and algorithm for drawing a rectifiable curve through the leaves of an infinite tree, which is inspired by P. Jones' proof of the TST
- "Necessary condition for 1-rectifiable" extends B-Schul (2015)
- Simpler formulation available if μ is pointwise doubling (drop lower density condition and replace $\beta_p^*(\mu, Q)$ by $\beta_p(\mu, 3Q)$)
- Open: Can we replace $\beta_p^*(\mu, Q)$ with $\beta_p(\mu, 3Q)$ for arbitrary μ?