Rectifiability of Measures:
the Identification Problem

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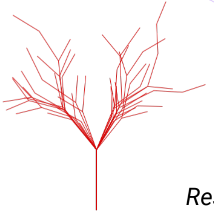
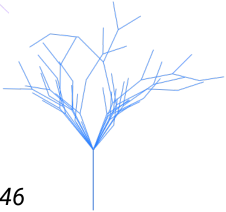


Interfaces between PDEs and GMT
Joint Mathematics Meetings
January 17, 2020



Research Partially Supported by NSF DMS 1650546

Some Open Ended Questions

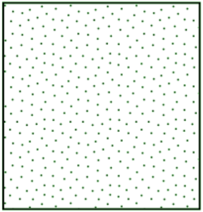
1. How can you describe a measure beyond talking about its null sets?

2. What does a generic measure look like?

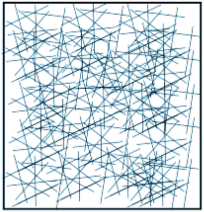
3. Can you decompose a complicated measure into simpler measures? Are there canonical ways to do it?

Preview: Three Measures

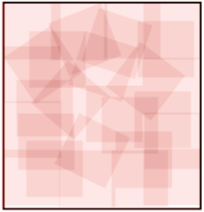
Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.
Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.



$\mu_0 = \sum_{i=1}^{\infty} a_i \, \delta_{x_i}$



$\mu_1 = \sum_{i=1}^{\infty} a_i \, L^1|_{\ell_i}$



$\mu_2 = \sum_{i=1}^{\infty} a_i \, L^2|_{S_i}$

- μ_0, μ_1, μ_2 are probability measures on $\mathbb{R}^2$
- The support of μ is the smallest closed set F carrying μ in the sense that $\mu(\mathbb{R}^2 \setminus F) = 0$; thus, $\operatorname{spt}\mu_0 = \operatorname{spt}\mu_1 = \operatorname{spt}\mu_2 = \mathbb{R}^2$
- μ_i is carried by i-dimensional sets (points, lines, squares)
- **The support of a measure is a rough approximation that hides the underlying structure of a measure**

Part I. Decomposition of Measures

Part II. Lipschitz Image Rectifiability

Part III. Fractional Rectifiability and Other Frontiers

Some Terminology (Missing from Standard Lexicon)

Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{N} \subset \mathcal{M}$ be non-empty.
Let $\mu : \mathcal{M} \to [0, \infty]$ be a measure.

- We say that μ is **carried by** $\mathcal{N}$ if there exists a sequence $N_1, N_2, \ldots$ of sets in $\mathcal{N}$ such that $\mu(X \setminus \bigcup_{i=1}^{\infty} N_i) = 0$.
- We say that μ is **singular to** $\mathcal{N}$ if $\mu(N) = 0$ for every $N \in \mathcal{N}$.

Lemma (decomposition)

If μ is σ-finite, then $\exists!$ σ-finite measures $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{N}}^{\perp}$ such that

$$\mu = \mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp},$$

where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$.

- This is an exercise in basic measure theory. The proof is sometimes embedded inside proofs of the Lebesgue-Radon-Nikodym theorem.
- The proof **does not** tell you how to find sets $N_1, N_2, \ldots$ that carry $\mu_{\mathcal{N}}$!!!

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The Identification Problem

For each

- measurable space $(X, \mathcal{M})$
- family $\mathcal{N} \subset \mathcal{M}$ of distinguished sets, and
- family $\mathscr{F}$ of σ-finite measures defined on $\mathcal{M}$,

the associated **identification problem** is to find pointwise properties that identify the part of μ carried by $\mathcal{N}$ and the part of μ singular to $\mathcal{N}$.

That is, find properties $P(\mu, x)$ and $Q(\mu, x)$ defined for all measures $\mu \in \mathscr{F}$ and all points $x \in X$ such that

- $\mu_{\mathcal{N}} = \mu \llcorner \{x \in X : P(\mu, x) \text{ holds}\}$ $\qquad \mu \llcorner A(B) = \mu(A \cap B)$
- $\mu_{\mathcal{N}}^{\perp} = \mu \llcorner \{x \in X : Q(\mu, x) \text{ holds}\}$

As in the Painlevé problem, the properties P and Q should depend on the geometry of the space X and the sets in $\mathcal{N}$.

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Interlude: How do you measure size of a set in $\mathbb{R}^n$?

Hausdorff measure and Hausdorff dimension

$$E \subset \mathbb{R}^n, \quad s \in [0, n]$$

$$\mathcal{H}^s(E) = \liminf_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} (\text{diam } E_i)^s : E \subseteq \bigcup_{i=1}^{\infty} E_i, \text{ where diam } E_i \leq \delta \right\}$$

$$\dim_{\mathcal{H}}(E) = s \text{ if and only if } \mathcal{H}^t(E) = \begin{cases} \infty & \text{when } t < s \\ 0 & \text{when } t > s \end{cases}$$

Packing measure and packing dimension

$$\mathcal{P}^s(E) = \inf \left\{ \sum_{i=1}^{\infty} P^s(E_i) : E \subseteq \bigcup_{i=1}^{\infty} E_i \right\},$$

$$P^s(E) = \limsup_{\delta \downarrow 0} \left\{ \sum_{i=1}^{\infty} (\text{diam } B_i)^s : B_i \text{ disjoint balls centered on } E, \text{diam } B_i \leq \delta \right\}$$

$$\dim_{\mathcal{P}}(E) = s \text{ if and only if } \mathcal{P}^t(E) = \begin{cases} \infty & \text{when } t < s \\ 0 & \text{when } t > s \end{cases}$$

$$\mathcal{H}^s(E) \leq \mathcal{P}^s(E), \quad \dim_{\mathcal{H}}(E) \leq \dim_{\mathcal{P}}(E)$$

Example: Measures Carried by / Singular to $\mathrm{Zero}(\mathcal{H}^s)$

Let $\mathrm{Zero}(\mathcal{H}^s)$ denote Borel sets in $\mathbb{R}^n$ of zero Hausdorff measure $\mathcal{H}^s$.

Decomposition Lemma: If μ is a σ-finite Borel measure on $\mathbb{R}^n$, then there exists a unique decomposition

$$\mu = \mu_{\mathrm{Zero}(\mathcal{H}^s)} + \mu^{\perp}_{\mathrm{Zero}(\mathcal{H}^s)}.$$

- $\mu_{\mathrm{Zero}(\mathcal{H}^s)}$ is carried by $\mathrm{Zero}(\mathcal{H}^s)$, i.e. $\mu_{\mathrm{Zero}(\mathcal{H}^s)} \perp \mathcal{H}^s$
- $\mu^{\perp}_{\mathrm{Zero}(\mathcal{H}^s)}$ is singular to $\mathrm{Zero}(\mathcal{H}^s)$, i.e. $\mu^{\perp}_{\mathrm{Zero}(\mathcal{H}^s)} \ll \mathcal{H}^s$

Identification for Radon measures: If μ is locally finite, then

- $\mu_{\mathrm{Zero}(\mathcal{H}^s)} = \mu \llcorner \left\{x \in \mathbb{R}^n : \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty\right\}$
- $\mu^{\perp}_{\mathrm{Zero}(\mathcal{H}^s)} = \mu \llcorner \left\{x \in \mathbb{R}^n : \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^s} < \infty\right\}$

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Theorem (Stratification by Upper and Lower Densities)

Let $\mathrm{Zero}(\mathcal{H}^s)$, $\mathrm{Finite}(\mathcal{H}^s)$, $\mathrm{Zero}(\mathcal{P}^s)$, $\mathrm{Finite}(\mathcal{P}^s)$ denote Borel sets in $\mathbb{R}^n$ of zero and finite Hausdorff measure $\mathcal{H}^s$ and packing measure $\mathcal{P}^s$.

Identification: If μ is a Radon measure on $\mathbb{R}^n$ and $s \in [0, n]$, then

- $\mu_{\mathrm{Zero}(\mathcal{H}^s)} = \mu \llcorner \left\{x \in \mathbb{R}^n : \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty\right\}$
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Recall that

$\mathcal{H}^s \leq \mathcal{P}^s$

Corollary (Stratification by Dimension)

Let μ be a non-zero σ-finite Borel measure on $\mathbb{R}^n$. There exists a countable set $\mathcal{S} = \mathcal{S}_{\mathcal{H}}(\mu) \subset [0, n]$ of dimensions, and for every $s \in \mathcal{S}$, there exists a unique non-zero σ-finite Borel measure μ_s such that

$$\mu = \sum_{s \in \mathcal{S}} \mu_s$$

and for each $s \in \mathcal{S}$ the measure μ_s is carried by Borel sets of Hausdorff dimension s and singular to Borel sets of Hausdorff dimension $t < s$.

Identification: Moreover, if μ is locally finite, then for each $s \in \mathcal{S}$,

$$\mu_s = \mu \llcorner \left\{ x \in \mathbb{R}^n : \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s-\epsilon}} = 0 \wedge \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{s+\epsilon}} = \infty \; \forall \epsilon > 0 \right\}.$$

- We may call $\mathcal{S}_{\mathcal{H}}(\mu)$ the **Hausdorff dimension spectrum** of μ.
- The lower/upper Hausdorff dimension of μ is $\inf \mathcal{S}_{\mathcal{H}}(\mu)$ / $\sup \mathcal{S}_{\mathcal{H}}(\mu)$.
- For every countable $S \subset [0, n]$, we can build μ with $\mathcal{S}_{\mathcal{H}}(\mu) = S$.
- Similar statements for packing dimension with lim inf instead of lim sup

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Lipschitz Images

A map $f : E \subset \mathbb{R}^m \to \mathbb{R}^n$ is **Lipschitz** if there exists a constant $0 \leq L < \infty$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in E$.

The infimal value of L in the inequality is attained and is called the **Lipschitz constant** of f, denoted by $\operatorname{Lip} f$.

- **Kirzbraun's Theorem:** There is a Lipschitz extension $F : \mathbb{R}^m \to \mathbb{R}^n$ of f with $\operatorname{Lip} F = \operatorname{Lip} f$.
- For any $F \subset E$, the restriction $g = f|_F$ of f to $F \subset E$ is Lipschitz with $\operatorname{Lip} g \leq \operatorname{Lip} f$.
- **Rademacher's Theorem:** f is differentiable at Lebesgue a.e. $x \in E$
- We may think of the image $f(E)$ as a "measure-theoretic" manifold, which admits tangent planes $\mathcal{H}^m$-a.e.

Exercise

If E is bounded, then $\mathcal{H}^m(f(E)) \leq \mathcal{P}^m(f(E)) \leq (\operatorname{Lip} f)^m \mathcal{P}^m(E) < \infty$.

In particular, if E is bounded, then $\mu = \mathcal{H}^m \llcorner f(E)$ and $\nu = \mathcal{P}^m \llcorner f(E)$ are finite measures on $\mathbb{R}^n$.

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If E is bounded, then $\mathcal{H}^m(f(E)) \leq \mathcal{P}^m(f(E)) \leq (\mathrm{Lip}\, f)^m \mathcal{P}^m(E) < \infty$.

In particular, if E is bounded, then $\mu = \mathcal{H}^m \llcorner f(E)$ and $\nu = \mathcal{P}^m \llcorner f(E)$ are finite measures on $\mathbb{R}^n$.

Lipschitz Images

A map $f : E \subset \mathbb{R}^m \to \mathbb{R}^n$ is **Lipschitz** if there exists a constant $0 \leq L < \infty$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in E$.

The infimal value of L in the inequality is attained and is called the **Lipschitz constant** of f, denoted by $\operatorname{Lip} f$.

- **Kirzbraun's Theorem:** There is a Lipschitz extension $F : \mathbb{R}^m \to \mathbb{R}^n$ of f with $\operatorname{Lip} F = \operatorname{Lip} f$.
- For any $F \subset E$, the restriction $g = f|_F$ of f to $F \subset E$ is Lipschitz with $\operatorname{Lip} g \leq \operatorname{Lip} f$.
- **Rademacher's Theorem:** f is differentiable at Lebesgue a.e. $x \in E$
- We may think of the image $f(E)$ as a "measure-theoretic" manifold, which admits tangent planes $\mathcal{H}^m$-a.e.

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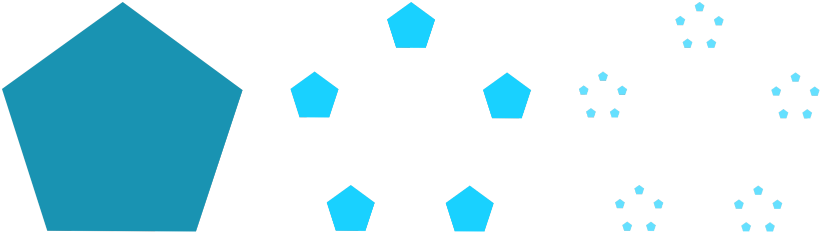
Self-similar Cantor Sets



Let $1 \leq m \leq n-1$ be integers.

Let $C \subset \mathbb{R}^n$ be a self-similar Cantor set of Hausdorff dimension m.

Theorem (Hutchinson 1981)

- *The measure* $\mu = \mathcal{H}^m \llcorner C$ *is finite and Ahlfors* m*-regular, i.e.* $\mu(B(x,r)) \approx r^m$ *for all* $x \in C$ *and* $0 < r \leq \operatorname{diam} C$.
- *The measure* μ *is singular to the set of Lipschitz images of* $\mathbb{R}^m$*, i.e.* $\mu(f(\mathbb{R}^m)) = 0$ *whenever* $f : \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz*

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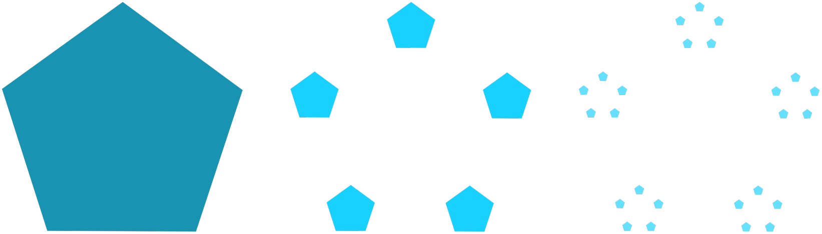

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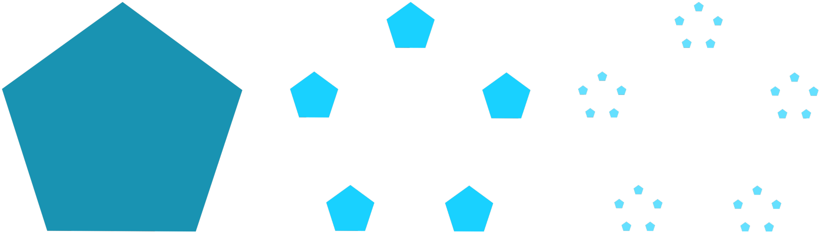

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Rectifiable curves and Cantor sets do not meet in a set of positive length

Rectifiable Measures (see Federer 1947 / 1969)

Let $\mathrm{Lip}(m, n)$ denote the set of images of Lipschitz maps $f : [0, 1]^m \to \mathbb{R}^n$. A Borel measure μ on $\mathbb{R}^n$ is called

- **(countably)** m**-rectifiable** if μ is carried by $\mathrm{Lip}(m, n)$
- **purely** m**-unrectifiable** if μ is singular to $\mathrm{Lip}(m, n)$

If μ is σ-finite, then there is a unique decomposition $\mu = \mu^m_{\text{rect}} + \mu^m_{\text{pu}}$, where μ^m_{rect} is m-rectifiable and μ^m_{pu} is purely m-unrectifiable.

Identification Problem: Find properties $P(\mu, x)$ and $Q(\mu, x)$ defined for all Radon measures μ on $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ such that

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Trivial when $m = n$. Solved when $m = 1$ and $n \geq 2$ (B-Schul 2017).
All other cases are open! (Do not assume $\mu \ll \mathcal{H}^m$)

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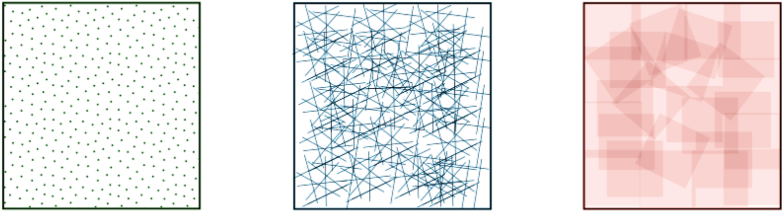
Reminder: Rectifiability is NOT about the Support

Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.

Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.



$\mu_0 = \sum_{i=1}^{\infty} a_i \, \delta_{x_i}$ $\quad$ $\mu_1 = \sum_{i=1}^{\infty} a_i \, L^1|_{\ell_i}$ $\quad$ $\mu_2 = \sum_{i=1}^{\infty} a_i \, L^2|_{S_i}$

- Supports are the same: $\operatorname{spt} \mu_0 = \operatorname{spt} \mu_1 = \operatorname{spt} \mu_2 = \mathbb{R}^2$
- μ_0 is 1-rectifiable and $\mu_0 \perp \mathcal{H}^1$
- μ_1 is 1-rectifiable and $\mu_1 \ll \mathcal{H}^1$
- μ_2 is purely 1-unrectifiable and $\mu_2 \perp \mathcal{H}^1$
- **The support of a measure is a rough approximation that hides the underlying structure of a measure**

Exercise: Rectifiable Measures and Lower Density

Every set $\Sigma \in \mathrm{Lip}(m, n)$ has $\mathcal{P}^m(\Sigma) < \infty$. Since μ^m_{rect} is carried by sets of finite packing measure, it follows that for locally finite μ,

$$\mu^m_{\mathrm{rect}} \leq \mu_{\mathrm{Finite}(\mathcal{P}^m)} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0 \right\}.$$

Thus, if μ is m-rectifiable, then $\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0$ μ-a.e.
(The converse is false; think $\mu = \mathcal{H}^m \llcorner C$ for self-similar Cantor sets)

Similarly,

$$\mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} = 0 \right\} = \mu^{\perp}_{\mathrm{Finite}(\mathcal{P}^s)} \leq \mu^m_{\mathrm{pu}}.$$

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Preiss' Theorem (Annals of Mathematics 1987)

Identification of m-rectifiable and purely m-unrectifiable parts of a Radon measure with finite upper density, i.e. singular to $\mathrm{Zero}(\mathcal{H}^m)$:

For all integers $1 \leq m \leq n-1$, there exists $c = c(m,n) < 1$ such that if μ is a Radon measure on $\mathbb{R}^n$ and $\limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$ μ-a.e., then

$$\mu^m_{\mathrm{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : 0 < \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} = \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} \right\}.$$

$$\mu^m_{\mathrm{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} \leq c \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} \right\}.$$

When $m=1$, $c = \frac{100}{101}$ [Morse-Randolph 1944, Moore 1950]

Proof Ingredients: **tangent measures**, uniform measures, moments, weak approximate tangent planes, Lipschitz graphs

Corollary: Let μ be a Radon measure on $\mathbb{R}^n$. Then μ is m-rectifiable and $\mu \ll \mathcal{H}^m$ iff $\lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m}$ exists and is > 0 and $< \infty$ μ-a.e.

Further Work on Absolutely Continuous Measures

Tolsa-Toro 2015: sufficient conditions for measures $\mu \ll \mathcal{H}^m$ to be m-rectifiable in terms of doubling defect

Tolsa 2015: necessary conditions for measures $\mu \ll \mathcal{H}^m$ to be m-rectifiable expressed in terms of Jones' beta numbers

Azzam-Tolsa 2015: sufficient conditions for measures $\mu \ll \mathcal{H}^m$ to be m-rectifiable in terms of Jones' beta numbers

Edelen-Naber-Valtorta arXiv 2016: sufficient conditions for measures $\mu \ll \mathcal{P}^m$ to be m-rectifiable in terms of Jones' beta numbers

Goering arXiv 2018: sufficient conditions for measures $\mu \ll \mathcal{H}^m$ to be m-rectifiable in terms of Menger type curvatures (cf. **Meurer 2018**)

Azzam-Tolsa-Toro arXiv 2018: sufficient conditions for pointwise doubling μ to be m-rectifiable and $\mu \ll \mathcal{H}^m$ in terms of Tolsa's alpha numbers

Dabrowski arXiv 2019: necessary and sufficient conditions for measures $\mu \ll \mathcal{H}^m$ to be m-rectifiable in terms of L^2 Wasserstein distances

Unilateral Linear Approximation Numbers

Let μ be a Radon measure on $\mathbb{R}^n$, let $Q \subset \mathbb{R}^n$ be a **window**, i.e. a bounded set of positive diameter, and let L be a m-dimensional plane. The **(non-homogeneous)** L^2 **Jones beta numbers** are

$$\beta_2^{(m)}(\mu, Q, L) := \left(\int_Q \left(\frac{\operatorname{dist}(x, L)}{\operatorname{diam} Q} \right)^2 \frac{d\mu(x)}{\mu(Q)} \right)^{1/2} \in [0, 1]$$

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- Jones 1990, David-Semmes 1991, 1993, Bishop-Jones 1994
- Non-homogeneous refers to scaling by $\mu(Q)$ to integrate against a probability measure
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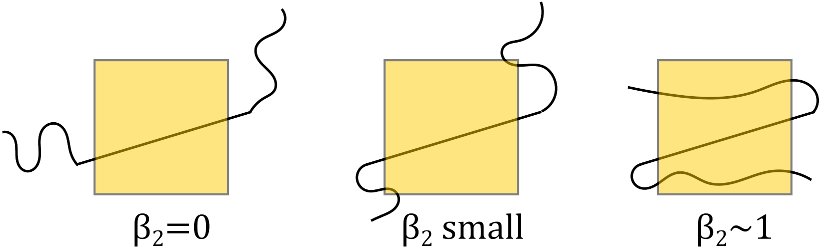
Unilateral Linear Approximation Numbers

Example Γ is a curve (black), measure $\mu = \mathcal{H}^1 \llcorner \Gamma$, dimension $m = 1$, window Q is a square (yellow)



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Theorem (B-Schul 2017 / Naples (forthcoming))

Let μ be a Radon measure on $\mathbb{R}^n$ or Hilbert space ℓ_2 and let $m = 1$. Assume μ is pointwise doubling, i.e. $\limsup_{r\downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$ *μ-a.e. Then*

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Let μ *be a Radon measure on* $\mathbb{R}^n$ *or Hilbert space* ℓ_2 *and let* $m = 1$*. Assume* μ *is pointwise doubling, i.e.* $\limsup_{r\downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty$ μ*-a.e. Then*

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- The first theorem identifies m-rectifiable part of Radon measures with $\mu \ll \mathcal{H}^m$ that are carried by sets of finite $\mathcal{H}^m$ measure
- The second theorem identifies 1-rectifiable part of Radon measures that are pointwise doubling ($\mu \perp \mathcal{H}^1$ allowed, no *a priori* dimension bound)
- Both theorems involve coarse densities, but in different ways

Failure to Characterize for Non-doubling Measures

The L^2 **density-normalized** Jones square function $\widetilde{J}_2$ is given by

$$\widetilde{J}_2(\mu, x) = \sum_Q \beta_2^{(1)}(\mu, 3Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) \in [0, \infty] \quad (x \in \mathbb{R}^n),$$

where Q ranges over all dyadic cubes in $\mathbb{R}^n$ of side length at most 1.

- If μ is 1-rectifiable, then $\widetilde{J}_2(\mu, x) < \infty$ μ-a.e. [B-Schul 2015]
- If μ is pointwise doubling and $\widetilde{J}_2(\mu, x) < \infty$ μ-a.e., then μ is 1-rectifiable.

Theorem (Martikainen-Orponen 2018)

For all $\varepsilon > 0$, there exists a Borel probability measure μ on $\mathbb{R}^2$ such that

- $\widetilde{J}_2(\mu, x) \leq \varepsilon$ *for all* $x \in \mathbb{R}^2$
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In particular, $\widetilde{J}_2(\mu, x) < \infty$ μ-a.e., but μ is purely 1-unrectifiable.

The enemy is the lack of (pointwise) doubling!

Non-doubling measures can “hide information” at coarse scales!!

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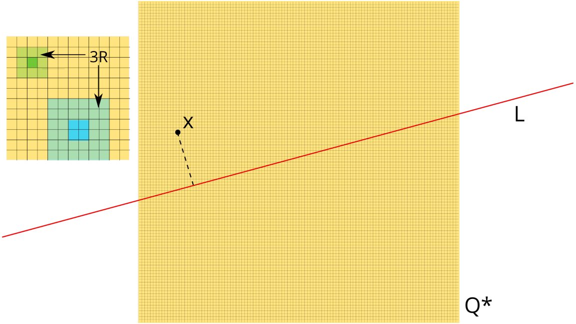
Anisotropic L^2 Jones β numbers (B-Schul 2017)

Given dyadic cube Q in $\mathbb{R}^n$, $\Delta^*(Q)$ denotes a subdivision of $Q^* = 1600\sqrt{n}Q$ into dyadic cubes R of same / previous generation as Q s.t. $3R \subseteq Q^*$.



For every Radon measure μ on $\mathbb{R}^n$ and every dyadic cube Q, we define $\beta_2^*(\mu, Q)^2 = \inf_{\text{line } L} \max_{R\in\Delta^*(Q)} \beta_2(\mu, 3R, L)^2 m_{3R}$, where

$$\beta_2(\mu, 3R, L)^2 m_{3R} = \int_{3R} \left(\frac{\operatorname{dist}(x, L)}{\operatorname{diam} 3R}\right)^2 \min\left(1, \frac{\mu(3R)}{\operatorname{diam} 3R}\right) \frac{d\mu(x)}{\mu(3R)}$$

Identification of 1-rectifiable and purely 1-unrectifiable parts of a measure in $\mathbb{R}^n$: a complete solution

Anisotropic L^2 density-normalized Jones function J_2^*:

$$J_2^*(\mu, x) = \sum_Q \beta_2^*(\mu, Q) \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) \in [0, \infty] \quad (x \in \mathbb{R}^n)$$

Theorem (B-Schul 2017)

If μ is a Radon measure on $\mathbb{R}^n$, then

$$\mu^1_{\mathrm{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r} > 0 \text{ and } J_2^*(\mu, x) < \infty \right\}$$

$$\mu^1_{\mathrm{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r} = 0 \text{ or } J_2^*(\mu, x) = \infty \right\}$$

New Ingredients: anisotropic beta numbers, technical extension of the Analyst's Traveling Salesman Theorem for point clouds

Scheme of the Proof in Three Steps

To solve the identification problem for locally finite measures on $(X, \mathcal{M})$ carried by / singular to $\mathcal{N}$...

1. Find a characterization of **subsets** of sets in $\mathcal{N}$

2. Convert the theorem for sets to a theorem for (pointwise) **doubling measures**

3. Introduce **anisotropic normalizations** to obtain a theorem for locally finite measures

Open Problem

Identification of m-rectifiable and purely m-unrectifiable parts of a measure:

When $2 \le m \le n-1$, find properties $P(\mu, x)$ and $Q(\mu, x)$ defined for all Radon measures μ on $\mathbb{R}^n$ such that

$$\mu^m_{\text{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} > 0 \text{ and } P(\mu, x) \text{ holds} \right\}$$

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- I expect the Harmonic Analysis and Geometric Measure Theory are now sufficiently well developed to solve this
- **Main Difficulty is Metric Geometry:** We lack a characterization of subsets of Lipschitz images $f([0,1]^m)$ in $\mathbb{R}^n$ when $2 \le m \le n-1$
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Part I. Decomposition of Measures

Part II. Lipschitz Image Rectifiability

Part III. Fractional Rectifiability and Other Frontiers

Grades of Rectifiability

Fractional Rectifiabilty	Integral Rectifiability	Higher-Order Rectifiability
real-valued dimensions	Lipschitz images of $[0,1]^m$, $m \geq 2$	$C^2, C^3, \ldots$
Hölder curves	bi-Lipschitz images	$C^{k,\alpha}$ graphs, $k \geq 1, 0 < \alpha < 1$
Hölder images of $[0,1]^m$, $m \geq 2$	Lipschitz graphs	Sobolev curves
bi-Hölder images	C^1 graphs	Sobolev graphs

Other Spaces Banach spaces, Carnot groups, manifolds, metric spaces

Lipschitz Image vs Lipschitz Graph vs C^1 Rectifiability

Theorem (see Federer 1969)

Let $1 \leq m \leq n-1$. If μ on $\mathbb{R}^n$ is Radon and $\mu \ll \mathcal{H}^m$, TFAE:

1. *μ is carried by Lipschitz images of $\mathbb{R}^m$, i.e. m-rectifiable*
2. *μ is carried by m-dimensional Lipschitz graphs* $1 \Rightarrow 2 \Rightarrow 3$ *trivial*
3. *μ is carried by m-dimensional C^1 graphs*

Theorem (Martín and Mattila 1988)

For all $0 < s < 1$, $\exists\, E, F \subset \mathbb{R}^2$ with $0 < \mathcal{H}^s(E), \mathcal{H}^s(F) < \infty$ such that

- *$\mu = \mathcal{H}^s \llcorner E$ is **carried by** Lipschitz images of $\mathbb{R}^1$, but*
- *μ is **singular to** 1-dimensional Lipschitz graphs;*
- *$\nu = \mathcal{H}^s \llcorner F$ is **carried by** 1-dimensional Lipschitz graphs, but*
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Lipschitz Graph Rectifiability via Cone Points

Let μ be a Radon measure on $X = \mathbb{R}^n$ or $X = \ell_2$ and let $1 \leq m < \dim(X)$ be an integer. Then $\mu = \mu_{\mathrm{LG}(m)} + \mu^{\perp}_{\mathrm{LG}(m)}$, where

- $\mu_{\mathrm{LG}(m)}$ carried by m-dimensional Lipschitz graphs,
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We may say x is an **m-dimensional cone point** for μ if there exists an m-dimensional cone X centered at x such that

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r) \setminus X))}{\mu(B(x,r))} = 0.$$

Theorem (Naples (forthcoming))

If μ is a pointwise doubling measure on X and $1 \leq m < \dim(X)$, then $\mu_{\mathrm{LG}(m)} = \mu \llcorner \{x \in X : x \text{ is an } m\text{-dimensional cone point for } \mu\}$

- Extends to pointwise doubling measures a classical theorem for measures with $0 < \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$ μ-a.e.
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Fractional Rectifiability

There are many dimensions between 1 and 2!

Idea (Martín and Mattila 1993): Use Hölder images to study rectifiability of sets / measures in non-integral dimensions

- For every $s \in [1,2]$, there is a four-corner Cantor set E_s with $0 < \mathcal{H}^s(E_s) < \infty$
- $\mathcal{H}^s \llcorner E_s$ is singular to $(1/s)$-Hölder curves
- $\mathcal{H}^s \llcorner E_s$ is carried by $(1/t)$-Hölder curves $\forall t > s$

Sufficient Conditions

Theorem (B-Vellis 2019; cf. Martín-Mattila 2000)

Let μ be a Radon measure on $\mathbb{R}^n$. If $m \leq s$ and $t < s$, then

$$\mu \llcorner \left\{ x \in \mathbb{R}^n : 0 < \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} < \infty \right\}$$

is carried by (m/s)-Hölder images of $[0,1]^m$.

Theorem (B-Naples-Vellis 2019)

Let μ be a pointwise doubling measure on $\mathbb{R}^n$. If $s \geq 1$, then

$$\mu \llcorner \left\{ x \in \mathbb{R}^n : \int_0^1 \beta_2^{(1)}(\mu, B(x,r))^2 \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty \right\}$$

is carried by $(1/s)$-Hölder curves.

- New **Hölder Traveling Salesman Theorem** giving a sufficient condition for a set to be contained inside a $(1/s)$-Hölder curve

Higher-Order Rectifiability

Theorem (see Federer 1969)

If $\mu = \mathcal{H}^m \llcorner E$, then E is carried by m-dimensional $C^{k,1}$ graphs if and only if E is carried by m-dimensional C^{k+1} graphs

Theorem (Anzelloti-Serapioni 1994)

*If $k + \alpha < l + \beta$, then there is a E on $\mathbb{R}^n$ such that $\mathcal{H}^m \llcorner E$ is carried by $C^{k,\alpha}$ graphs **and** singular to $C^{l+\beta}$ graphs*

- Anzelloti-Serapioni characterized $C^{1,\alpha}$ rectifiability of measures $\mu = \mathcal{H}^m \llcorner E$.
- Generalized to higher-orders by Santilli 2019

A Jones-type Sufficient Condition

Theorem (Ghinassi arXiv 2017)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $1 \leq m \leq n-1$. Assume that $0 < \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$ μ-a.e. For all $0 < \alpha < 1$,

$$\mu \llcorner \left\{ x : \int_0^1 \frac{\beta_2^{(m)}(\mu, B(x,r))^2}{r^{2\alpha}} \frac{\mu(B(x,r))}{r^m} \frac{dr}{r} < \infty \right\}$$

is carried by $C^{1,\alpha}$ graphs.

- A similar result holds at $\alpha = 1$
- Higher-orders and necessity are open

Rectifiability in Other Metric Spaces

There are many challenges, partial results, and
open directions

A partial list of work...

Preiss and Tišer 1992

Kircheim 1994

Cheeger 1999

Leger 1999

Ambrosio and Kircheim 2000

Lorent 2003, 2004, 2007

Mattila, Serapioni, Serra Cassano 2010

Bate 2015

Chousionis and Tyson 2015

Bate, Csörnyei, Wilson 2017

Bate-Li 2017

Chousionis, Fässler, and Orponen 2019

David-Schul 2019

B-McCurdy forthcoming

B-Li-Zimmerman forthcoming

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Thank you for listening!