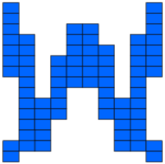
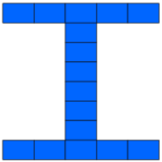
Hölder parameterizations of Bedford-McMullen carpets and connected IFS





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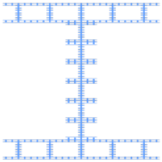

Fall AMS Meeting Madison
A&P on Metric Spaces and Fractals
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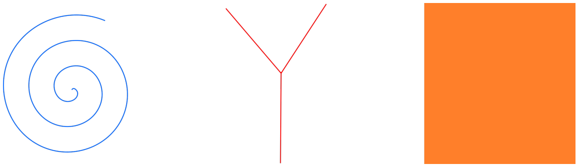
Joint work with Vyron Vellis (Tennessee)

This research is partially supported by NSF grant DMS 1650546

What is a curve?

A **curve** Γ in a metric space X is a **continuous image** of $[0,1]$:
There exists a continuous map $f : [0,1] \to X$ such that $\Gamma = f([0,1])$

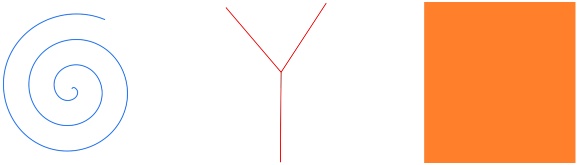


A continuous map f with $\Gamma = f([0,1])$ is called a **parameterization** of Γ

- There are curves which do not have a 1-1 parameterization
- There are curves which have topological dimension > 1
- The modulus of continuity of a parameterization is a proxy for the size/regularity/complexity of a curve

Two characterizations (early 20th century)

Hahn-Mazurkiewicz Theorem A set Γ in a metric space is a **curve** if and only if Γ is compact, connected, locally connected.



Ważewski Theorem A set Γ in a metric space is a **rectifiable curve** iff Γ is a **Lipschitz curve** iff Γ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$

- In $\mathbb{R}^n$: Lipschitz curves (also called *rectifiable curves*) admit unique tangent lines at $\mathcal{H}^1$-a.e. point by Rademacher's theorem
- Note compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$ implies Γ locally connected! This can fail for sets with σ-finite length (e.g. a topologist's comb)

f is Lipschitz if $\exists L < \infty$ such that $|f(x) - f(y)| \leq L|x - y|$ for all x, y

$\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure

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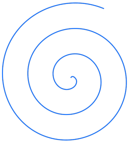 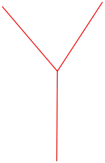 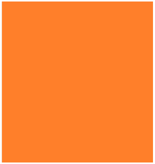

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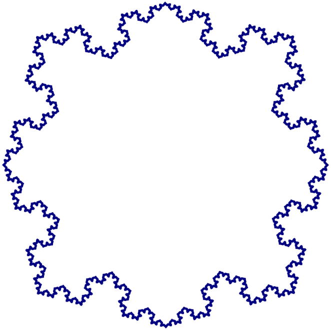
What about higher-dimensional curves?



What about higher-dimensional curves?

Open Problem (#1)

For each real $s \in (1, \infty)$, characterize curves $\Gamma \subset \mathbb{R}^n$ with $\mathcal{H}^s(\Gamma) < \infty$

Open Problem (#2)

*For each real $s \in (1, \infty)$, characterize $(1/s)$-**Hölder curves**, i.e. sets that can be presented as $h([0, 1])$ for some map $h : [0, 1] \to \mathbb{R}^n$ with*

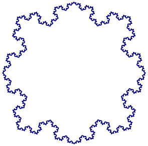
$$|h(x) - h(y)| \leq H|x - y|^{1/s}$$

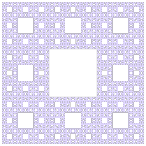
- Every $(1/s)$-Hölder curve has $\mathcal{H}^s(\Gamma) < \infty$ (exercise)
- Example (B-Naples-Vellis, Adv. Math. 2019): For every $s \in (1, n)$, $\exists$ a curve $\Gamma \subset \mathbb{R}^n$ that is s-**Ahlfors regular,** $\mathcal{H}^s(\Gamma \cap B(x, r)) \approx r^s$, but Γ is **not** a $(1/s)$-Hölder curve.

Why?

- There are many dimensions between 1 and 2
- Lipschitz surfaces are Hölder curves:
  to study the former, first study the latter
- Martín and Mattila (1993,2000) developed a portion of Besicovitch's fine theory of 1-sets in $\mathbb{R}^2$ works for s-sets in $\mathbb{R}^n$ using Hölder curves as a replacement for rectifiable curves
- There exist metric spaces without rectifiable curves that are Hölder path connected
- Modulus of path families makes sense for Hölder curves
- Random settings: Brownian motion, rough paths theory
- Possible tool for singular integrals in codimension > 1

Sufficient conditions for Hölder curves





Theorem (Remes 1998)

*Let $S \subset \mathbb{R}^n$ be a **self-similar set** satisfying the **open set condition**.*
If S is connected, then S is a $(1/s)$-Hölder curve, $s = \dim_H S$.

A set $E \subset \mathbb{R}^n$ is β-**flat** if for every $x \in E$ and $r > 0$, there exists a line ℓ such that $\mathrm{dist}(x, \ell) \leq \beta r$ for all $x \in E \cap B(x, r)$.

Theorem (B-Naples-Vellis, Adv. Math. 2019)

*There exists a universal constant $\beta_0 \in (0, 1)$ such that if $E \subset \mathbb{R}^n$ is β_0-**flat**, connected, compact, $\mathcal{H}^s(E) < \infty$, and $\mathcal{H}^s(E \cap B(x, r)) \gtrsim r^s$, then E is a $(1/s)$-Hölder curve.*

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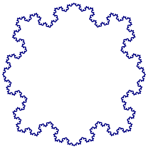

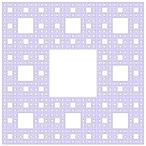

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Iterated Function Systems (Quick Review)

Let X be a complete, separable metric space. A **contraction** in X is a map $\phi : X \to X$ with Lipschitz constant $\mathrm{Lip}(\phi)$ strictly less than 1

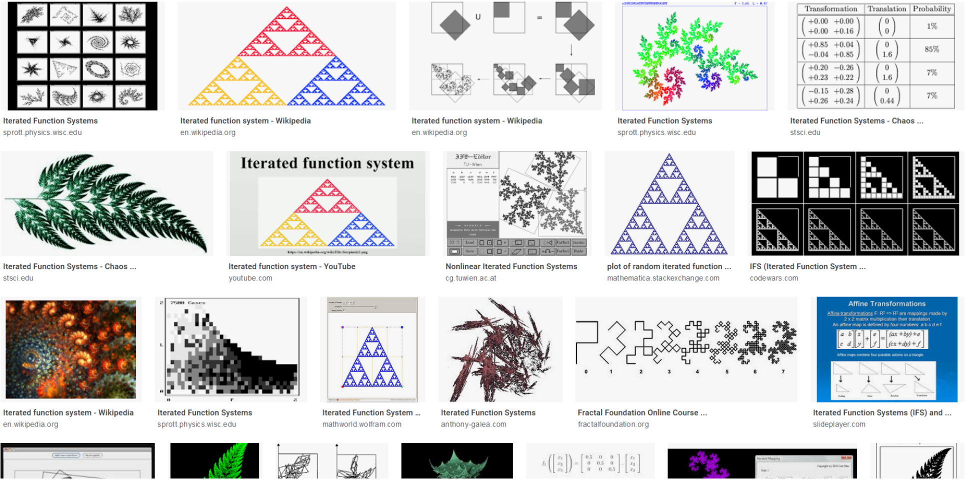
$$\mathrm{Lip}(\phi) = \inf\{L \geq 0 : \mathrm{dist}(\phi(x), \phi(y)) \leq L\, \mathrm{dist}(x, y)\}$$

Hutchinson's Theorem For every finite family $\mathcal{F}$ of contractions in X, there exists a unique compact set $K \subset X$ such that $K = \bigcup_{\phi \in \mathcal{F}} \phi(K)$.

- $\mathcal{F}$ is called an **iterated function system**
- $K = K_{\mathcal{F}}$ is called the **attractor** of $\mathcal{F}$
- $\mathcal{H}^s(K) < \infty$ where s is the **similarity dimension** of $\mathcal{F}$, i.e.

$$s \geq 0 \text{ is the unique number such that } \sum_{\phi \in \mathcal{F}} \mathrm{Lip}(\phi)^s = 1$$

- If each $\phi \in \mathcal{F}$ is a similarity, i.e. $\mathrm{dist}(\phi(x), \phi(y)) = \lambda_\phi\, \mathrm{dist}(x, y)$, then we call K a **self-similar set**



”Iterated Function Systems”, Google Image Search, 3pm (Atlanta) on 9/13/2019

IFS with Connected Attractors

Theorem (Hata 1985)

*Let $\mathcal{F}$ be an IFS over a complete metric space. If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is **path connected** and **locally connected**. Thus, $K_{\mathcal{F}}$ is a **curve**.*

Let $\mathcal{F}$ be an IFS over a complete metric space; let s be the similarity dimension of $\mathcal{F}$.

Theorem (B-Vellis, arXiv October 2019 (I'm optimistic))

*If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is $(1/s)$-**Hölder path connected**.*

Theorem (B-Vellis, arXiv October 2019)

*If $K_{\mathcal{F}}$ is connected, then $K_{\mathcal{F}}$ is a $(1/\alpha)$-**Hölder curve** for every $\alpha > s$.*

- Second theorem is a corollary of the first, viewing $K_{\mathcal{F}}$ as leaves of a tree with $(1/s)$-Hölder edges (cf. B-Vellis JGA 2019)

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Remes' parameterization of self-similar sets

Let $\mathcal{F}$ be an IFS over a complete metric space X that is **generated by similarities**; let s be the similarity dimension of $\mathcal{F}$.

Theorem

*If $K_{\mathcal{F}}$ is connected **and** $\mathcal{H}^s(K_{\mathcal{F}}) > 0$, then $K_{\mathcal{F}}$ is a $(1/s)$-Hölder curve.*

- Remes (1998) proved this when $X = \mathbb{R}^n$, where $\mathcal{H}^s(K_{\mathcal{F}}) > 0 \Leftrightarrow SOSC \Leftrightarrow OSC \Rightarrow \dim_H K_{\mathcal{F}} = s$ (Schief 1994)
- In complete metric spaces: $\mathcal{H}^s(K_{\mathcal{F}}) > 0 \Rightarrow SOSC \Rightarrow \dim_H K_{\mathcal{F}} = s$ (Schief 1996)
- Self-similar implies $K_{\mathcal{F}}$ with $\mathcal{H}^s(K_{\mathcal{F}}) > 0$ are s-Ahlfors regular
- To prove thm, embed $K_{\mathcal{F}}$ in ℓ_∞ and repeat Remes' original proof

Open Set Condition: $\exists U$ open s.t. $\phi(U) \subseteq U$, $\phi(U) \cap \psi(U) = \emptyset$ for distinct $\phi, \psi \in \mathcal{F}$. Strong Open Set Condition: also $U \cap K_{\mathcal{F}} \neq \emptyset$.

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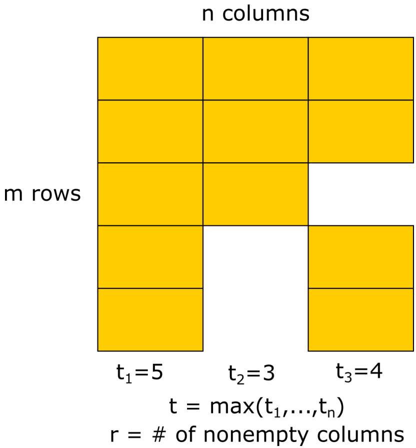
Case Study: Bedford-McMullen Carpets



Let Σ be a Bedford-McMullen carpet (see diagram).

- Similarity dimension is $s = \log_n(t_1 + \cdots + t_n)$

McMullen (1984)

- $\dim_H \Sigma = \log_n \left(\sum_{j=1}^n t_j^{\log_m(n)} \right)$
- $\dim_M \Sigma = \log_n(r) + \log_m \left(\sum_{j=1}^n t_j / r \right)$

Mackay (2011)

- If $m < n$ (self-affine), then $\dim_A \Sigma = \log_n(r) + \log_m(t)$

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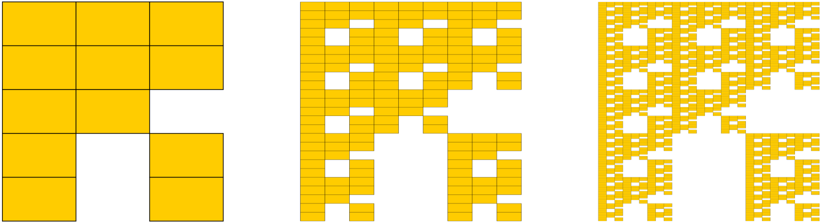

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Theorem (B-Vellis arXiv October 2019)

Let $\Sigma \subset [0,1]^2$ be a connected Bedford-McMullen carpet.

- *If Σ is a line, Σ is (trivially) a 1-Hölder curve*
- *If Σ is the square, Σ is (well-known to be) a $(1/2)$-Hölder curve*
- *Otherwise, Σ is a $(1/s)$-Hölder curve, s similarity dimension*

The exponents are sharp (they cannot be increased).

- Idea: **Lift** Σ to a self-similar set K in $([0,1]^2, d)$ equipped with a partially snowflaked metric d via a **Lipschitz map** $F : K \to \Sigma$. Use Remes' theorem upstairs to parameterize K. Then descend.
- If X doubling: $\mathcal{H}^s(K) > 0 \Leftrightarrow SOSC$ (Stella 1992 / Schief 1996)
- When does an IFS admit a Lipschitz lift to self-similar set in doubling space (or a β-space)?

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Self-similar vs self-affine carpets and a conjecture

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D = Hausdorff dimension, s = similarity dimension (sometimes $s > 2$)!

Self-Similar

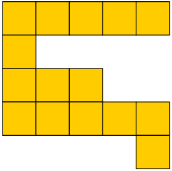
$D = s$, $s \in [1, 2]$

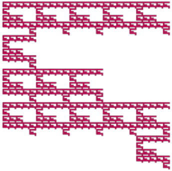
$0 < \mathcal{H}^s(\Sigma) < \infty$

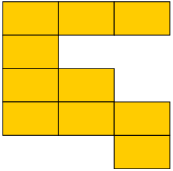
Self-Affine

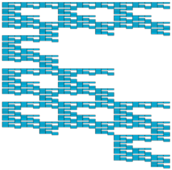
$D < s$, $s \in [1, \infty)$

$\mathcal{H}^s(\Sigma) = 0$









Conjecture (B 2018): If $\Gamma \subset \mathbb{R}^n$ is a $(1/s)$-Hölder curve with $\mathcal{H}^s(\Gamma) > 0$, then at $\mathcal{H}^s$-a.e. $x \in \Gamma$, all geometric blow-ups (tangent sets) of Γ at x are "self-similar" $(1/s)$-Hölder images of $\mathbb{R}$

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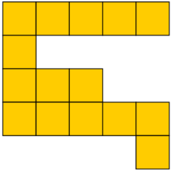
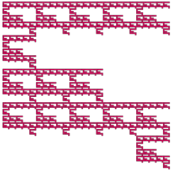
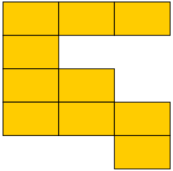
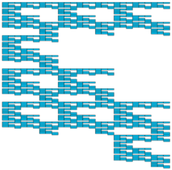
$D < s$, $s \in [1, \infty)$

$\mathcal{H}^s(\Sigma) = 0$









Conjecture (B 2018): If $\Gamma \subset \mathbb{R}^n$ is a $(1/s)$-Hölder curve with $\mathcal{H}^s(\Gamma) > 0$, then at $\mathcal{H}^s$-a.e. $x \in \Gamma$, all geometric blow-ups (tangent sets) of Γ at x are “self-similar” $(1/s)$-Hölder images of $\mathbb{R}$

Related and Future Work

A related, but different problem: What sets in a metric space are contained in a $(1/s)$-Hölder curve?

- Hölder Traveling Salesman Theorem in $\mathbb{R}^n$ (sufficient conditions): B-Naples-Vellis Adv. Math. 2019
- New result in quasiconvex metric spaces: Balogh and Züst arXiv 2019 (in summer)

Future work:

- We need to find good necessary conditions for a set to be (contained in) a $(1/s)$-Hölder curve
- Applications to Geometry of Measures (cf. **Lisa Naples**' talk), Metric Geometry. Random geometry? Singular integrals?

Thank you for listening!