Square Packings and Rectifiable Doubling Measures

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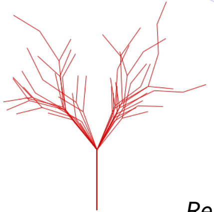

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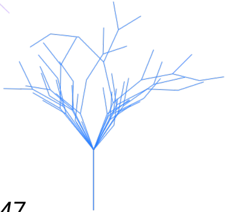

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Part I **Background and Related Results** (not comprehensive)

Part II "Square Packing Construction" of Lipschitz Maps

Part III Rectifiable Doubling Measures in Ahlfors Regular Spaces

Lipschitz Images and Rectifiability

Let $\mathbb{M}$ and $\mathbb{X}$ be metric spaces.

Think of $\mathbb{M}$ as the **model space** and $\mathbb{X}$ as the **target space**.

Lipschitz image problem For which sets $F \subset \mathbb{X}$, does there exist a Lipschitz map $f : \mathbb{M} \to \mathbb{X}$ such that $F = f(\mathbb{M})$?

Lipschitz fragment problem For which sets $F \subset \mathbb{X}$, does there exist a set $E \subset \mathbb{M}$ and a Lipschitz map $f : E \to \mathbb{X}$ such that $F = f(E)$?

Rectifiable set problem For which sets $F \subset \mathbb{X}$, does there exist a sequence of Lipschitz maps $f_i : E_i \subset \mathbb{M} \to \mathbb{X}$ such that $F = \bigcup_1^\infty f_i(E_i)$?

Rectifiable measure problem For which Radon measures μ on $\mathbb{X}$, is there a sequence of Lipschitz maps such that $\mu(\mathbb{X} \setminus \bigcup_1^\infty f_i(E_i)) = 0$?

Rectifiable Curves and Analyst's Traveling Salesman

$\mathbb{M} = [0, 1]$: a Lipschitz image $f([0, 1])$ is called a **rectifiable curve**.

Theorem (Ważewski 1927)

In any metric space $\mathbb{X}$, a set $F \subset \mathbb{X}$ is a rectifiable curve if and only if F is compact, F is connected, and Hausdorff measure $\mathcal{H}^1(F) < \infty$.

Theorem (Jones-Okikiolu-Schul-Badger-McCurdy 1990–2023)

In any finite-dimensional Banach space or in any Hilbert space $\mathbb{X}$, a set $F \subset \mathbb{X}$ is contained in a rectifiable curve if and only if

$$\operatorname{diam} F < \infty \quad \text{and} \quad \sum_Q \beta_F^2(Q) \operatorname{diam} Q < \infty,$$

where Q ranges over an appropriate family of "dyadic locations and scales" and $\beta_F(Q)$ measures how close $F \cap 3Q$ is to a line relative to $\operatorname{diam} Q$.

Theorem (Li 2022, earlier work by Ferrari-Franchi-Pajot, ...)

$\exists$ *characterization of subsets of rectifiable curves in Carnot groups of step ≥ 2*

Characterization of 1-Rectifiable Measures in Euclidean Spaces and Carnot Groups

A Radon measure μ on $\mathbb{X}$ is **1-rectifiable** in the sense of Federer if there exist Lipschitz $f_i : E_i \subset [0,1] \to \mathbb{X}$ such that $\mu(\mathbb{X} \setminus \bigcup_1^\infty f_i(E_i)) = 0$.

Theorem (Badger-Schul 2017)

Let $\mathbb{X} = \mathbb{R}^d$ for some $d \geq 2$. A Radon measure μ is 1-rectifiable iff

$$\liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r} > 0 \quad \textit{and} \quad \sum_Q \beta_\mu^*(Q)^2 \,\mathrm{diam}\, Q \frac{\chi_Q(x)}{\mu(Q)} < \infty \text{ at } \mu\text{-a.e. } x,$$

*where $\beta_\mu^*(Q)$ is an **anisotropic beta number** associated to $\mu \llcorner 1600Q$.*

Theorem (Badger-Li-Zimmerman 2023)

Let $\mathbb{X} = \mathbb{G} =$ Carnot group of step $s \geq 2$. A Radon measure μ is 1-rectifiable iff

$$\liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r} > 0 \quad \textit{and} \quad \sum_Q \beta_\mu^*(Q)^{2s} \,\mathrm{diam}\, Q \frac{\chi_Q(x)}{\mu(Q)} < \infty \text{ at } \mu\text{-a.e. } x,$$

*where $\beta_\mu^*(Q)$ is a **stratified anisotropic beta number** associated to $\mu \llcorner 1600Q$.*

Characterization of Rectifiable Curve Fragments in Arbitrary Metric Spaces

In a metric space $\mathbb{X}$, define

$$Z(x_1, \ldots, x_n) = \max \left\{ \sum_{j=1}^{k-1} |x_{i_j} - x_{i_{j+1}}| : 1 = i_1 < \cdots < i_k = n \right\},$$

$$\delta(F) = \sup \left\{ \min_{\pi \in S_n} Z(x_{\pi(1)}, \ldots, x_{\pi(n)}) : x_1, \ldots, x_n \in F, n \geq 1 \right\}$$

Theorem (Balka-Keleti arXiv-2023)

Let $\mathbb{X}$ be a metric space and let $F \subset \mathbb{X}$ be compact. There exists a compact set $E \subset [0, 1]$ such that $F = f(E)$ for some Lipschitz map $f : E \subset [0, 1] \to \mathbb{X}$ if and only if $\delta(F) < \infty$.

My Interpretation: Modify definition of total variation, realizing that we don't know *a priori* the correct order to visit all of the points in F.

Quick Review of Minkowski and Packing Dimensions

Let $\mathbb{X}$ be a metric space. The **(upper) Minkowski dimension** or **(upper) box counting dimension** of a bounded set $F \subset \mathbb{X}$ is

$$\limsup_{r \downarrow 0} \frac{\log(\text{minimum number of balls of radius } r \text{ needed to cover } F)}{\log r}$$

Unfortunately it is possible that $\dim_M(\bigcup_1^\infty F_i) \neq \sup_1^\infty \dim_M F_i$, i.e. Minkowski dimension is not countably stable.

Example: In $\mathbb{X} = \mathbb{R}$, $\dim_M\{1/n : n \geq 1\} = 1/2$, but $\sup_{n \geq 1} \dim_M\{1/n\} = 0$.

The **(upper) packing dimension** of a set $F \subset \mathbb{X}$ is

$$\inf\{\sup \dim_M F_i : F = \bigcup_1^\infty F_i,\ F_i \text{ bounded}\}.$$

This is equivalent to another well-known definition with packing measures, but I don't need those today. In general $\dim_H F \leq \dim_P F \leq \dim_M F$.

A "Universal" Sufficient Condition for Higher-Dimensional Lipschitz Fragments

Theorem (Balka-Keleti arXiv-2023)

Suppose $\mathbb{M}$ is compact and has Hausdorff dimension t. If $F \subset \mathbb{X}$ is compact and the Minkowski dimension of F is $< t$, then $F = f(E)$ for some Lipschitz map $f : E \subset \mathbb{M} \to \mathbb{X}$.

Proof Ingredients Combine Balka and Keleti's new characterization of rectifiable curve fragments with two theorems from metric geometry:

- Mendel and Naor's ultrametric skeleton theorem (2013) and
- Keleti-Máthé-Zindulka's theorem (2014) on existence of Lipschitz surjections from ultrametric spaces onto $[0, 1]^m$.

Corollary

If $F \subset \mathbb{X}$ has packing dimension $< m$, then F is an m-rectifiable set in the sense that $F \subset \bigcup_1^\infty f_i(E_i)$ for some Lipschitz maps $f_i : E_i \subset [0, 1]^m \to \mathbb{X}$.

Open Problem: Lipschitz Images of Squares into $\mathbb{R}^3$

Let $\mathbb{M} = [0,1]^2$ be a Euclidean square.

Let $\mathbb{X} = \mathbb{R}^3$ be a 3-dimensional Euclidean space.

Lipschitz image problem For which sets $F \subset \mathbb{R}^3$, does there exist a Lipschitz map $f : [0,1]^2 \to \mathbb{R}^3$ such that $F = f([0,1]^2)$?

Lipschitz fragment problem For which sets $F \subset \mathbb{R}^3$, does there exist a Lipschitz map $f : [0,1]^2 \to \mathbb{R}^3$ such that $F \subset f([0,1]^2)$. Equivalent to original formulation by McShane's extension theorem.

Rectifiable set problem Because of translation invariance of $\mathbb{R}^3$, this is likely equivalent to the Lipschitz fragment problem.

Rectifiable measure problem For which Radon measures μ on $\mathbb{R}^3$, are there Lipschitz maps $f_i : [0,1]^2 \to \mathbb{R}^3$ s.t. $\mu(\mathbb{R}^3 \setminus \bigcup_1^\infty f_i([0,1]^2)) = 0$?

Partial Results for m-Rectifiable Measures

A Radon measure μ on $\mathbb{X}$ is **m-rectifiable** in the sense of Federer if there exist Lipschitz $f_i : E_i \subset [0,1]^m \to \mathbb{X}$ such that $\mu(\mathbb{X} \setminus \bigcup_1^\infty f_i(E_i)) = 0$.

Theorem (Morse-Randolph-Moore-Preiss 1944–1987)

A Radon measure μ on $\mathbb{R}^d$ is m-rectifiable if

$$0 < \lim_{r \downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

Theorem (Corollary of Balka-Keleti arXiv-2023)

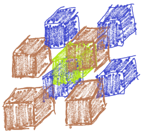
A Radon measure μ on a metric space $\mathbb{X}$ is m-rectifiable if

$$\limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} < m \quad \text{at } \mu\text{-a.e. } x \in \mathbb{X}.$$

Consequence: To characterize 2-rectifiable measures in $\mathbb{R}^3$, it remains to understand the rectifiability of sets in $\mathbb{R}^3$ that simultaneously have (i) zero Hausdorff measure $\mathcal{H}^2$ and (ii) packing dimension 2.

A 2-dimensional null set in $\mathbb{R}^3$ that is not a distorted copy of a null set in $\mathbb{R}^2$

Start with $F_0 = [0,1]^3$. Assume F_n has been defined and consists of 9^n cubes with mutually disjoint interiors and side length s_n. Define F_{n+1} by replacing each cube Q in F_n with 9 cubes of side length $s_{n+1} = \frac{1}{n+1}3^{-(n+1)}$, eight in the corners and one in the center. Then $F = \bigcap_{n=0}^{\infty} F_n$ is Cantor set.



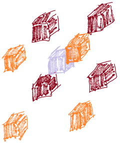
It is easy to see that $\mathcal{H}^2(F) = 0$ and $\dim_H F = \dim_P F = \dim_M F = 2$.

Theorem: F is not contained in a Lipschitz image of $[0,1]^2$.

Different (unpublished) proofs communicated to me and Raanan Schul by David-Toro (2016) and Alberti-Csörnyei (2019)

A 2-dimensional null set in $\mathbb{R}^3$ that is a distorted copy of a null set in $\mathbb{R}^2$

Let $\alpha > 1$. Modify the side lengths so that $s_{n+1} = \frac{1}{(n+1)^\alpha} 3^{-(n+1)}$.



Once again $\mathcal{H}^2(F) = 0$ and $\dim_H F = \dim_P F = \dim_M F = 2$.

Theorem: There exists a compact set $E \subset [0,1]^2$ and a Lipschitz map $f : E \to \mathbb{R}^3$ such that $F = f(E)$.

An (unpublished) construction of this type was found by Badger-Vellis (2019). More systematic proof is given in Badger-Schul (2023-arXiv).

Part I Background and Related Results

Part II “Square Packing Construction” of Lipschitz Maps

Part III Rectifiable Doubling Measures in Ahlfors Regular Spaces

Combinatorial Problem: Square Packing

Problem: Suppose you are given a list of side lengths

$$s_0 > s_1 > s_2 > \cdots > s_{n-1}.$$

What is the side length $\mathrm{side}(s_0, \ldots, s_{n-1})$ of the smallest square containing squares of side length $s_0, \ldots, s_{n-1}$ with disjoint interiors?

Theorem (Moon-Moser 1967)

$\mathrm{side}(s_0, \ldots, s_{n-1})^2 \leq 2 \sum_{i=0}^{n-1} s_i^2$.

Remark 1: Taking $s_0 = s_1 = 1$ and $s_2 = s_3 \ll 1$ shows that the multiplicative factor 2 in the lemma is sharp.

Remark 2: The multiplicative factor 2 in the lemma is deadly for iterative constructions. This gives a heuristic explanation of why the diam^2 gauge ("diameter squared") has not lead to a 2d traveling salesman theorem.

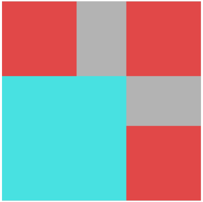
Remark 3: When packing intervals (1d squares), the corresponding statement is much nicer: $\mathrm{side}(s_0, \ldots, s_{n-1}) = s_0 + \cdots + s_{n-1}$.

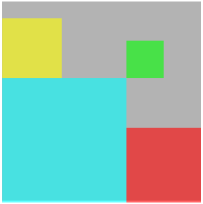
Diameter-Based Square Packing Bound

Lemma (Badger-Schul arXiv-2023)

$\text{side}(s_0, \ldots, s_{n-1}) \leq s_0 + s_1 + s_4 + s_9 + \ldots$ *(add squared indices only)*

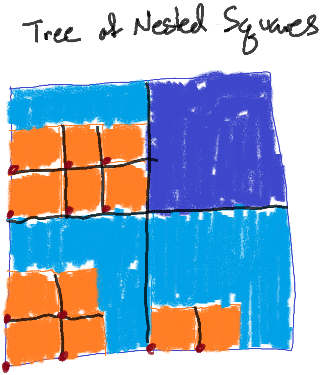
Proof. When $n = 4$, the smallest square containing squares of side length s_0, s_1, s_2, and s_3 has side length $s_0 + s_1$.

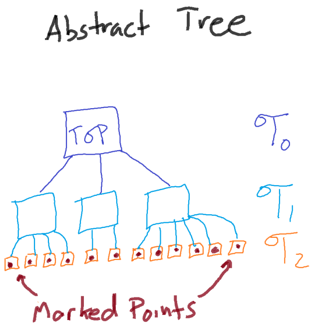




Corollary (Restatement of an Obvious Fact): A list of N squares of side length s can be packed inside of a square of side length $\lceil N^{1/2} \rceil s$

Idea: Represent a tree of nested sets in $\mathbb{X}$ as a combinatorially equivalent tree of nested squares in $\mathbb{R}^2$





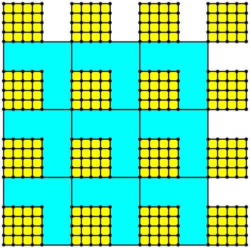
Pick a level $l \geq 1$. Given marked points $\{x_Q : Q \in \mathcal{T}_l\}$, need to decide how to place a points $x'_Q = f^{-1}(x_Q)$ in the domain such that

$$|x_Q - x_R| = |f(x'_Q) - f(x'_R)| \leq |x'_Q - x'_R| \quad \text{or} \quad |x'_Q - x'_R| \geq |x_Q - x_R|.$$

Recursive construction (outline)

1. Suppose we can do the construction for trees of depth $l-1$.
2. Let $\mathcal{T}$ be a tree of depth $l \geq 1$ and let $\{x_Q : Q \in \mathcal{T}_l\}$ be given.
3. View the tree $\mathcal{T}$ as a disjoint union of $N_0 = \#\,\mathrm{Child}(\mathrm{Top}(\mathcal{T})) = \#\mathcal{T}_1$ trees of depth $l-1$ (one such tree for each set in $\mathcal{T}_1$). For each $P \in \mathcal{T}_1$, we let $F_P = \{x_Q : Q \in \mathcal{T}_l \text{ and } Q \text{ is a descendant of } P\}$.
4. For each $P \in \mathcal{T}_1$, we can find a square $S_P \subset \mathbb{R}^2$, "domain points" $E_P \subset S_P$, and an 1-Lipschitz bijective map $f_P : E_P \to F_P$.
5. Key step: Let $D_0 = \mathrm{diam}\,\mathrm{Top}(\mathcal{T})$ and let $s_{l-1} = \max\{\mathrm{side}\, S_P : P \in \mathcal{T}_1\}$. Use the Lemma to pack $(\lceil N_0^{1/2} \rceil - 1)^2$ cubes of side length $s_{l-1} + D_0$ and $\lceil N_0^{1/2} \rceil^2 - ((\lceil N_0^{1/2} \rceil - 1)^2$ cubes of side length s_{l-1} into a cube of side
$$s_l = (\lceil N_0^{1/2} \rceil - 1)(D_0 + s_{l-1}) + s_{l-1} = \lceil N_0^{1/2} \rceil s_{l-1} + (\lceil N_0^{1/2} \rceil - 1) D_0.$$
6. Solve the recursion formula.

Example of the domain of a map with level $l = 2$



There are 16 cubes in $\mathcal{T}_1$ and each cube in $\mathcal{T}_1$ has 25 children.

"Yellow blocks" are translations of squares produced by the recursive step. The side length of a "yellow block" is $(\lceil N_1^{1/2} \rceil - 1)D_1 = 4D_1$.

To get a 1-Lipschitz map, we must surround each "yellow block" (except the rightmost ones in each direction) by a "blue" gap of side length D_0.

The total side length of the big square is

$$(\lceil N_0^{1/2} \rceil - 1)D_0 + \lceil N_0^{1/2} \rceil (\lceil N_1^{1/2} \rceil - 1)D_1 = 3D_0 + 4 \cdot 4D_1 = 3D_0 + 16D_1$$

Square Packing Construction

Theorem (Badger-Schul arXiv-2023)

Let $\mathcal{T} = \bigsqcup_{j=0}^{\infty} \mathcal{T}_j$ be a tree of sets in a metric space $\mathbb{X}$, requiring only that every set in the tree is contained in its parent. For each $j \geq 0$, assign

$$N_j = \max_{Q \in \mathcal{T}_j} \#\mathrm{Child}(Q) \quad \textit{and} \quad D_j = \max_{Q \in \mathcal{T}_j} \operatorname{diam} Q.$$

Let $l \geq 1$ be an integer and suppose that $\mathcal{T}_l \neq \emptyset$. Compute

$$s = \sum_{j=0}^{l-1} \left(\prod_{i=0}^{j-1} \lceil N_i^{1/2} \rceil \right) (\lceil N_j^{1/2} \rceil - 1) D_j.$$

(When $j = 0$, $\prod_{i=0}^{j-1} \lceil N_i^{1/2} \rceil = 1$.) For any set or multiset $F = \{x_Q \in Q : Q \in \mathcal{T}_l\}$, there is a set $E \subset \ell_\infty^2 \cap [0, s]^2$ with $\#E = \#F$ and we can construct a 1-Lipschitz bijection $f : E \to F$.

Remark (Hölder maps): If you replace the quantity D_j in s by D_j^α, then the construction produces Hölder bijection $f : E \to F$ of exponent $1/\alpha$.

Square Packing Construction + Arzela-Ascoli

Corollary (Badger-Schul arXiv-2023)

Let $\mathcal{T}$ be a tree of nested sets in a metric space $\mathbb{X}$, requiring only that every set in the tree is contained in its parent. Assume each level of the tree is nonempty. As before, for each $j \geq 0$, assign

$$N_j = \max_{Q \in \mathcal{T}_j} \#\mathrm{Child}(Q) \quad \textit{and} \quad D_j = \max_{Q \in \mathcal{T}_j} \mathrm{diam}\, Q.$$

Suppose that

$$L = \sum_{j=0}^{\infty} \left(\prod_{i=0}^{j} \lceil N_i^{1/2} \rceil \right) D_j < \infty.$$

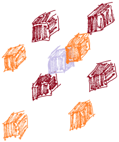
Then there exists a compact set $E \subset [0,1]^2$ and an L-Lipschitz map $f : [0,1]^2 \cap \ell_\infty^2 \to \mathbb{X}$ such that $\mathrm{Leaves}(\mathcal{T}) \subset f(E)$.

Remark (Higher-Dimensional Domains): The same construction lets you build Lipschitz maps from subsets of $[0,1]^m$ when $m \geq 3$. Simply replace the quantity $\lceil N_j^{1/2} \rceil$ with $\lceil N_j^{1/m} \rceil$.

Example of Using the Square Packing Construction

Let $\alpha > 1$. Recall that we built a Cantor set F in $\mathbb{R}^3$ by starting with $[0, 1]^3$ and then replacing each cube in level n with 9 children of side length $s_{n+1} = \frac{1}{(n+1)^\alpha} 3^{-(n+1)}$.

Cubes in level n have $N_n = 9$ children and diameter $D_n = \frac{\sqrt{3}}{n^\alpha} 3^{-n}$



$$L = \sum_{n=0}^{\infty} \left(\prod_{k=0}^{n} \lceil N_k^{1/2} \rceil \right) D_n = \sum_{n=0}^{\infty} 3^{n+1} D_n = 3\sqrt{3} + 3\sqrt{3} \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty$$

Therefore, there exists a compact set $E \subset [0, 1]^2$ an L-Lipschitz map $f : E \to \mathbb{R}^3$ such that $f(E) = F$.

Easy Application of Square Packing Construction

The Assouad dimension of a set in a metric space (I'll skip the definition) is at least as large as the Minkowski dimension: $\dim_M F \leq \dim_A F$.

For any $b > 1$ the definition of $\dim_A F$ naturally yields a tree of sets with $\text{Leaves}(\mathcal{T}) = F$ and $N_j = \max\{\#\,\text{Child}(Q) : Q \in \mathcal{T}_j\} \lesssim b^{\dim_A F}$ and $D_j = \max\{\text{diam}\, Q : Q \in \mathcal{T}_j\} \leq b^{-j}$. Taking b to be sufficiently large and applying the Square Packing Construction to $\mathcal{T}$ gives

Theorem (Badger-Schul arXiv-2023)

If $\mathbb{X}$ is a complete metric space. If $F \subset \mathbb{X}$ is compact, $m \geq 1$ is an integer, and $\dim_A F < m$, then $\exists$ Lipschitz map $f : E \subset [0,1]^m \to \mathbb{X}$ such that $f(E) \supset F$.

Remark 1: This theorem is now superseded by Balka-Keleti.

Remark 2: On the other hand, the proof of Assouad dimension theorem is much easier/shorter than the proof of the Minkowski dimension theorem.

Remark 3: Balka-Keleti is not specific to Euclidean domains and cannot be used to check m-rectifiability for sets of dimension m (like previous slide).

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Part III Rectifiable Doubling Measures in Ahlfors Regular Spaces

Context: Doubling Measures and Rectifiable Curves

For this talk a doubling measure μ on a metric space $\mathbb{X}$ is a Radon measure such that for some constant C,

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty \quad \text{for all } x \in \mathbb{X} \text{ and } r > 0$$

Theorem (Volberg-Konyagin 1987, Luukkainen-Saskman 1998)

If $\mathbb{X}$ is a complete, doubling metric space (i.e. every ball of radius $2r$ can be covered by at most C' balls of radius r), then there are doubling measures on $\mathbb{X}$.

Lemma

If μ is a doubling measure on $\mathbb{R}^d$, $d \geq 2$, and $F \subset \mathbb{R}^d$ is q-Ahlfors regular set for some $q < d$, then F is porous and $\mu(F) = 0$. In particular, doubling measures on $\mathbb{R}^d$ do not charge C^1 and bi-Lipschitz curves (i.e. they are μ null sets).

Theorem (Garnett-Killip-Schul 2010)

For all $d \geq 2$, there exist doubling measures μ on $\mathbb{R}^d$ that are 1-rectifiable. Hence $\mu(\Gamma) > 0$ for some rectifiable curve Γ (with Assouad dimension d).

Rectifiable Doubling Measures with Prescribed Hausdorff and Packing Dimensions

Theorem (Badger-Schul arXiv-2023)

Let $\mathbb{X}$ be a complete, Ahlfors q-regular metric space. Let m be an integer with $q > m - 1$. Given any $0 < s_H < s_P < q$ with $m - 1 < s_P < m$ and $s_P < q$, there exists a doubling measure μ on $\mathbb{X}$ such that

1. *μ has Hausdorff dimension s_H: $\liminf_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = s_H$ at μ-a.e. x,*
2. *μ has packing dimension s_P: $\limsup_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = s_P$ at μ-a.e. x,*
3. *μ is m-rectifiable (i.e. carried by Lipschitz images of $E \subset [0,1]^m$),*
4. *μ is purely $(m-1)$-unrectifiable*
   (i.e. singular to Lipschitz images of $E \subset [0,1]^{m-1}$).

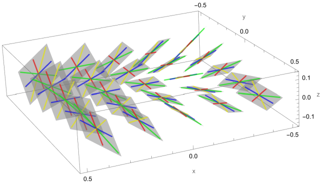
Conjecture (Badger-Schul arXiv-2023): The theorem also holds at the endpoint parameters, i.e. if $s_P = m - 1$, $s_P = m$, or $s_P = q$.

Examples

1. There exist doubling measures μ on $\mathbb{R}^3$ of Hausdorff dimension $s_H = 0.0001$ and packing dimension $s_P = 1.9999$ that are 2-rectifiable and purely 1-unrectifiable.
2. Any compact self-similar set of Hausdorff dimension q in $\mathbb{R}^d$ that satisfies the open set condition is Ahlfors q-regular and supports a $\lceil q \rceil$-rectifiable doubling measure that is purely $(\lceil q \rceil - 1)$-unrectifiable. These examples include Cantor sets, which are totally disconnected.
3. The Koch snowflake curve in $\mathbb{R}^2$ contains no non-trivial rectifiable subcurves, but is Ahlfors $\log_3(4)$-regular. Thus, the snowflake curve supports 1-rectifiable doubling measures of Hausdorff and packing dimension $1 - \epsilon$ for any $\epsilon > 0$.
4. When $s > m$ and $I = [0, 1]^m$ is equipped with the *snowflake metric* $d(x, y) = |x - y|^{m/s}$, the space I is Ahlfors s-regular and $\mathcal{H}^s \llcorner I$ is purely m-unrectifiable (because $s > m$). Nevertheless, the space I supports an m-rectifiable doubling measure that is purely $(m - 1)$-unrectifiable.

"Euclidean-Like" Measures on the Heisenberg Group

The first Heisenberg group $\mathbb{H}^1$ is a nonabelian step 2 Carnot group that is topologically equivalent to $\mathbb{R}^3$, but equipped with a metric so that $\mathbb{H}^1$ has Hausdorff dimension 4 and is Ahlfors 4-regular.



By theorem of Ambrosio and Kirchheim, the Hausdorff measures $\mathcal{H}^m \llcorner \mathbb{H}^1$ are purely m-unrectifiable for all $m \in \{2, 3, 4\}$. Even so, for all $m \in \{2, 3, 4\}$ and $s < m$, there exist doubling measures μ on $\mathbb{H}^1$ and Lipschitz maps $f : E \subset \mathbb{R}^m \to \mathbb{H}^1$ such that $\mu \ll \mathcal{H}^{s-\epsilon}$ for all $\epsilon > 0$, $\dim_H f(E) = s$, and $\mu(f(E)) > 0$. That is, **doubling measures on $\mathbb{H}^1$ can charge Lipschitz images of Euclidean spaces of almost maximal dimension.**

Remarks

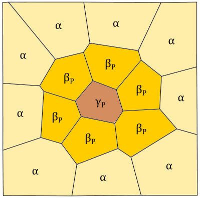
- $s_P > m - 1$ implies that μ is purely $(m-1)$-unrectifiable is (or should be!) well-known
- Any doubling measure on vanishes on porous sets, including images of lower-dimensional bi-Lipschitz embeddings into $\mathbb{R}^d$. So a bi-Lipschitz technique like David and Toro's variant of the Reifenberg algorithm is useless for proving rectifiability of a doubling measure.
- To prove that μ is m-rectifiable, we use $s_P < m$ and the square packing construction, but this also follows from Balka-Keleti. Finer analysis with square packing construction should yield the case $s_P = m$.
- In these examples, Hausdorff dimension is essentially irrelevant to rectifiability. It is packing dimension that matters.
- So the essential point is to build a doubling measure μ satisfying $m - 1 < \dim_P \mu < m$. For $\mathbb{X} = \mathbb{R}^d$, we can use a Bernoulli product. For general Ahlfors regular $\mathbb{X}$, we could not locate such measures in the literature and build quasi-Bernoulli measures using the metric cubes of Käenmäki-Rajala-Suomala

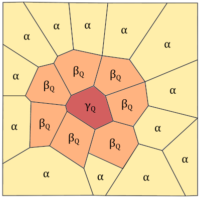
Quasi-Bernoulli Measures with Prescribed Dimensions

Let $\mathbb{X}$ be complete, q-Ahlfors regular. We start with any doubling measure ν, pick a sequence $\mathbf{s} = (s_k)_{k=1}^{\infty}$ of "target dimensions", let Δ be a system of (KRS) b-adic cubes on $\mathbb{X}$ with $b \geq 47$ sufficiently large depending on $\mathbb{X}$ and $\mathbf{s}$, and then redistribute the mass below scale 1 to produce a doubling measure μ with prescribed dimensions: $\dim_H \mu = \liminf_{n\to\infty} \frac{1}{n}(s_1 + \cdots + s_n)$ and $\dim_P \mu = \limsup_{n\to\infty} \frac{1}{n}(s_1 + \cdots + s_n)$

Without exact counts of cubes, we need to do some actual work to arrange that the entropy of each level of the system takes prescribed values.

To get a doubling measure and prescribed entropy we need three weights per cube. It is crucial that the "outer weights" α do not depend on the cube Q.





The Key Computation: How to Pick the Weights

Lemma (Badger-Schul arXiv-2023)

If $b > 1$ and L and M are positive integers such that $L \leq b^y$ and $M \geq b^s$ for some $s, y > 0$, then there exists a number $\alpha_0 = \alpha_0(b, y, s)$ such that for all $0 \leq \alpha \leq \alpha_0$, there exist unique numbers $\beta = \beta(\alpha, b, y, s, L, M)$ and $\gamma = \gamma(\alpha, b, y, s, L, M)$ such that

$$L\alpha + (M-1)\beta + \gamma = 1 \tag{1}$$

and the entropy function

$$h_{b,L,M}(\alpha, \beta) := L\alpha \log_b(1/\alpha) + (M-1)\beta \log_b(1/\beta) + \gamma \log_b(1/\gamma) = s. \tag{2}$$

We may always bound $L\alpha \log_b(1/\alpha) \leq \min(1,s)/e$, $L\alpha \leq \min(1,s^2)/e^2$,

$$\gamma \geq 1 - L\alpha - \frac{s - L\alpha \log_b(1/\alpha)}{\log_b(M-1)} \geq 1 - \frac{\min(1,s^2)}{e^2} - \left(1 - \frac{1}{e}\right)\frac{s}{\log_b(M-1)}, \tag{3}$$

$$\text{and} \qquad \gamma \geq \frac{1 - L\alpha}{M} \geq \frac{1}{M}\left(1 - \frac{\min(1,s^2)}{e^2}\right) \geq \frac{1}{M}\left(1 - \frac{1}{e^2}\right). \tag{4}$$

Moreover, if $2e^2 \log_b(e^2) \leq (\frac{1}{2} - \frac{1}{e})s$, then

$$\beta \geq \frac{s}{2(M-1)\log_b(M-1)}. \tag{5}$$

Actual Definition of the Quasi-Bernoulli Measures

Definition (Badger-Schul arXiv-2023)

Let $\mathbb{X}$ be a complete Ahlfors q-regular metric space with $\operatorname{diam}\mathbb{X} \geq 2.1$, let ν be a doubling measure on $\mathbb{X}$, and let $\mathbf{s} = (s_k)_{k=1}^{\infty}$ be a sequence of positive numbers ("target dimensions") such that

$$s_* := \inf_{k\geq 1} s_k > 0 \quad \text{and} \quad s^* := \sup_{k\geq 1} s_k < q. \tag{6}$$

Let $(\Delta_k)_{k\in\mathbb{Z}}$ be a system of b-adic cubes for $\mathbb{X}$ for some large $b \geq 47$. For all $Q \in \Delta$, assign $L_Q := \#\mathrm{Outer}(Q)$, $M_Q := \#\mathrm{Inner}(Q)$, and $N_Q := \#\,\mathrm{Child}(Q)$. We require that b be large enough depending on at most $\mathbb{X}$ and s^* so that

$$M_Q \geq b^{s^*} \quad \text{and} \quad L_Q \leq N_Q \leq b^{q+1} \quad \text{for all } Q \in \Delta_+ = \bigcup_{k=0}^{\infty} \Delta_k. \tag{7}$$

Let $0 < \alpha \leq \left(\frac{1}{2}\min\{s_*, 1\}\ln(b) b^{-(q+1)}\right)^2$ be a given weight. For all $k \geq 0$ and $Q \in \Delta_k$, use the Lemma to define unique weights

$$\beta_Q = \beta(\alpha, b, q+1, s_{k+1}, L_Q, M_Q) \quad \text{and} \quad \gamma_Q = \gamma(\alpha, b, q+1, s_{k+1}, L_Q, M_Q)$$

satisfying

$$1 = L_Q\alpha + (M_Q - 1)\beta_Q + \gamma_Q \quad \text{and} \quad h_{b,L_Q,M_Q}(\alpha, \beta_Q) = s_{k+1}. \tag{8}$$

We specify a Radon measure $\mu_{\mathbf{s}}$ on $\mathbb{X}$ by specifying its values on cubes as follows:

1. Declare $\mu_{\mathbf{s}}(Q) := \nu(Q)$ for all $Q \in \Delta_0$.
2. For all $k \geq 0$ and $Q \in \Delta_k$, declare $\mu_{\mathbf{s}}(R) := \alpha\mu_{\mathbf{s}}(Q)$ for all $R \in \mathrm{Outer}(Q)$, declare $\mu_{\mathbf{s}}(R) := \beta_Q\mu_{\mathbf{s}}(Q)$ for all $R \in \mathrm{Inner}(Q) \setminus \{Q^{\downarrow}\}$, and declare $\mu_{\mathbf{s}}(Q^{\downarrow}) := \gamma_Q\mu_{\mathbf{s}}(Q)$.

We call $\mu_{\mathbf{s}}$ a *quasi-Bernoulli measure on* $\mathbb{X}$ *with target dimensions* $\mathbf{s}$, *background measure* ν, and *outer weight* α.


Thank you for your attention!
View from the UConn Math Department