Structure theorems for Radon measures

Matthew Badger

Department of Mathematics
University of Connecticut

Analysis on Metric Spaces
University of Pittsburgh
March 10–11, 2017

Research Partially Supported by NSF DMS 1500382, 1650546

Decomposition of Measures

Geometric Measure Theory: *Understand a measure on a space through its interaction with canonical sets in the space.*

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} \subseteq \mathcal{M}$ be a family of measurable sets.

- μ is carried by $\mathcal{N}$ if there exist countably many sets $\Gamma_i \in \mathcal{N}$ such that $\mu\left(X \setminus \bigcup_i \Gamma_i\right) = 0$.
- μ is singular to $\mathcal{N}$ if $\mu(\Gamma) = 0$ for every $\Gamma \in \mathcal{N}$.

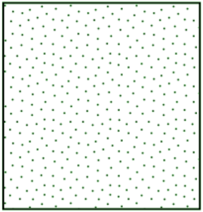
Exercise (Decomposition Lemma)

If μ is σ-finite, then μ can be written uniquely as $\mu_{\mathcal{N}} + \mu_{\mathcal{N}}^{\perp}$ where $\mu_{\mathcal{N}}$ is carried by $\mathcal{N}$ and $\mu_{\mathcal{N}}^{\perp}$ is singular to $\mathcal{N}$.

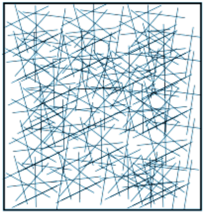
- Proof of the Decomposition Theorem is abstract nonsense.
- **Identification Problem**: Find measure-theoretic and/or geometric characterizations or constructions of $\mu_{\mathcal{N}}$ and $\mu_{\mathcal{N}}^{\perp}$?

PSA: Don't Think About Support

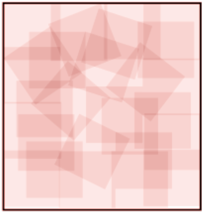
Three Measures. Let $a_i > 0$ be weights with $\sum_{i=1}^{\infty} a_i = 1$.
Let $\{x_i : i \geq 1\}$, $\{\ell_i : i \geq 1\}$, $\{S_i : i \geq 1\}$ be a dense set of points, unit line segments, unit squares in the plane.



$\mu_0 = \sum_{i=1}^{\infty} a_i \delta_{x_i}$



$\mu_1 = \sum_{i=1}^{\infty} a_i L^1 \llcorner \ell_i$



$\mu_2 = \sum_{i=1}^{\infty} a_i L^2 \llcorner S_i$

- μ_0, μ_1, μ_2 are probability measures on $\mathbb{R}^2$
- spt μ is smallest closed set carrying μ;
  spt μ_0 = spt μ_1 = spt μ_2 = $\mathbb{R}^2$
- μ_i is carried by i-dimensional sets (points, lines, squares)
- **The support of a measure is a rough approximation that hides the underlying structure of a measure**

Example: Atomic Measures vs Atomless Measures

A **Radon measure** is a locally finite, Borel regular outer measure.

Let μ be a Radon measure on $\mathbb{R}^n$. Then we can write

$$\mu = \mu^0_{\text{rect}} + \mu^0_{\text{pu}},$$

where

- μ^0_{rect} is carried by singletons (i.e. μ^0_{rect} is atomic)
- μ^0_{pu} is singular to singletons (i.e. μ^0_{pu} is atomless)

How can you identify μ^0_{rect} and μ^0_{pu}?

$$\mu^0_{\text{rect}} = \mu \llcorner \{x \in \mathbb{R}^n : \lim_{r\downarrow 0} \mu(B(x,r)) > 0\}$$

$$\mu^0_{\text{pu}} = \mu \llcorner \{x \in \mathbb{R}^n : \lim_{r\downarrow 0} \mu(B(x,r)) = 0\}$$

$\mu \llcorner E$ denotes the restriction of μ to E: $(\mu \llcorner E)(F) = \mu(E \cap F)$

1-Rectifiable and Purely 1-Unrectifiable Measures

A singleton $\{x\}$ has the following properties:

- $\{x\}$ is connected, compact, $\mathcal{H}^0(\{x\}) < \infty$

A candidate Γ for a "1-dimensional atom" might satisfy:

- Γ is connected, compact, $\mathcal{H}^1(\Gamma) < \infty$

Theorem

A set $\Gamma \subseteq \mathbb{R}^n$ is connected, compact, and $\mathcal{H}^1(\Gamma) < \infty$ if and only if there exists a Lipschitz map $f\colon [0,1] \to \mathbb{R}^n$ such that $\Gamma = f([0,1])$.

A connected, compact set $\Gamma \subseteq \mathbb{R}^n$ with $\mathcal{H}^1(\Gamma) < \infty$ is called a **rectifiable curve** or **Lipschitz curve**

Decomposition: Let μ be a Radon measure on $\mathbb{R}^n$. Then

$$\mu = \mu^1_{\text{rect}} + \mu^1_{\text{pu}},$$

where

- μ^1_{rect} is carried by rectifiable curves (μ^1_{rect} is 1-rectifiable)
- μ^1_{pu} is singular to rectifiable curves (μ^1_{pu} is purely 1-unrectifiable)

Identification Problem for μ^1_{rect} and μ^1_{pu}

If $E \subseteq \mathbb{R}^n$ and $\mathcal{H}^1(E) < \infty$, then

$$\frac{1}{2} \leq \limsup_{r \downarrow 0} \frac{\mathcal{H}^1(E \cap B(x,r))}{2r} \leq 1 \quad \mathcal{H}^1\text{-a.e. } x \in E.$$

Solution for Hausdorff measures:

Theorem (Besicovitch 1928,1938)

Let $\mu = \mathcal{H}^1 \llcorner E$, where $E \subseteq \mathbb{R}^2$ and $0 < \mathcal{H}^1(E) < \infty$. Then

$$\mu^1_{\text{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^2 : \lim_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} = 1 \right\}$$

$$= \mu \llcorner \left\{ x \in \mathbb{R}^2 : E \text{ has a tangent line at } x \right\}$$

$$\mu^1_{\text{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^2 : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} \leq \frac{3}{4} \right\}$$

$$= \mu \llcorner \left\{ x \in \mathbb{R}^2 : E \text{ does not have a tangent line at } x \right\}$$

Identification Problem for μ^1_{rect} and μ^1_{pu}

Let μ be a Radon measure on $\mathbb{R}^n$. The following are equivalent:

- $\mu \ll \mathcal{H}^1$ (i.e. $\mathcal{H}^1(E) = 0 \Longrightarrow \mu(E) = 0$)
- $\limsup_{r\downarrow 0} \dfrac{\mu(B(x,r))}{2r} < \infty$ at μ-a.e. $x \in \mathbb{R}^n$

Solution for absolutely continuous measures:

Theorem (Morse and Randolph 1944)

Let μ be a Radon measure on $\mathbb{R}^2$ with $\mu \ll \mathcal{H}^1$. Then

$$\mu^1_{\text{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^2 : 0 < \lim_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} < \infty \right\}$$

$$= \mu \llcorner \left\{ x \in \mathbb{R}^2 : E \text{ has a } \mu\text{-approximate tangent line at } x \right\}$$

$$\mu^1_{\text{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^2 : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} \leq \frac{1}{1.01} \limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} \right\}$$

$$= \mu \llcorner \left\{ x \in \mathbb{R}^2 : E \text{ does not have a } \mu\text{-approximate tangent line at } x \right\}$$

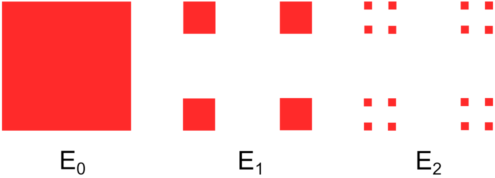
Examples I

- For $1 \leq i \leq k$, let E_i be a subset of a rectifiable curve $\Gamma_i \subseteq \mathbb{R}^n$. $\mu = \sum_{i=1}^{k} \mathcal{H}^1 \llcorner E_i$ is a finite 1-rectifiable measure, $\mu \ll \mathcal{H}^1$.

- Let C be the Cantor middle-third set and let $\mu = \mathcal{H}^s \llcorner C \times \{0\}$, where $s = \log(2)/\log(3)$ is the Hausdorff dimension of C. μ is 1-rectifiable measure with $\mu(B(x,r)) \sim r^s$ for all $x \in \operatorname{spt} \mu$ and $0 < r \leq 1$; and $\mu \perp \mathcal{H}^1$ and $\limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{2r} = \infty$ μ-a.e.

Examples II

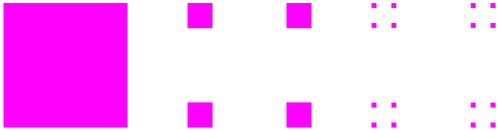
- Lebesgue measure on $\mathbb{R}^n$ is purely 1-unrectifiable for all $n \geq 2$
  (This is obvious!)
- Let $E \subseteq \mathbb{R}^2$ be the "4 corners" Cantor set, $E = \bigcap_{i=0}^{\infty} E_i$



- E is Ahlfors regular: $\mathcal{H}^1(E \cap B(x,r)) \sim r$ for all $x \in E$, $0 < r \leq 1$.
- Every rectifiable curve $\Gamma = f([0,1]) \subset \mathbb{R}^2$ intersects E in a set of zero $\mathcal{H}^1$ measure.
- $\mathcal{H}^1 \llcorner E$ is a purely 1-unrectifiable measure on $\mathbb{R}^2$

Examples III

Let $E_\lambda \subseteq \mathbb{R}^2$ be the generalized "4 corners" Cantor set, where $0 < \lambda \leq 1/2$ is the scaling factor.



$(\lambda = 1/5)$

- E has Hausdorff dimension $s = \log(4)/\log(1/\lambda)$
- $\mathcal{H}^s(E_\lambda \cap B(x, r)) \sim r^s$ for all $x \in E_\lambda$ and $0 < r \leq 1$.
- When $\lambda = 1/2$, $s = 2$ and $\mathcal{H}^s \llcorner E$ is just Lebesgue measure restricted to the unit square.
- If $1/4 \leq \lambda \leq 1/2$, then $\mathcal{H}^s \llcorner E_\lambda$ is purely 1-unrectifiable
- If $0 < \lambda < 1/4$, then $\mathcal{H}^s \llcorner E_\lambda$ is 1-rectifiable
  see e.g. Martin and Mattila (1988)

Examples IV

Theorem (Garnett-Killip-Schul 2010)

There exist a Radon measure μ on $\mathbb{R}^2$ with spt $\mu = \mathbb{R}^2$ *such that μ is* ***doubling*** *($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$), $\mu \perp \mathcal{H}^1$, and μ is 1-rectifiable.*

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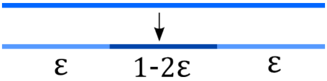

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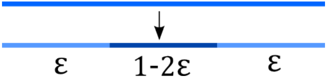

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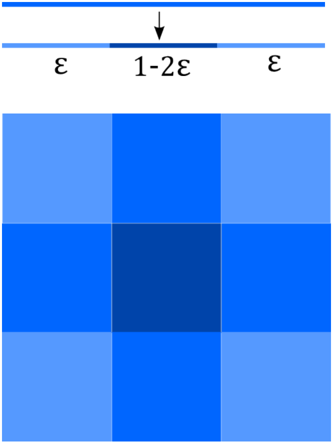

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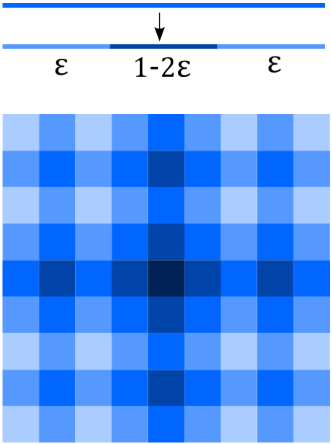

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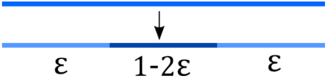

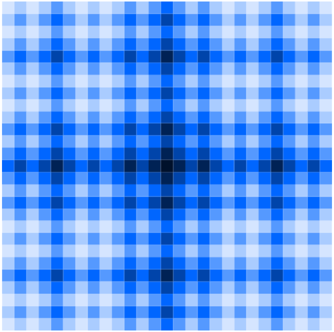

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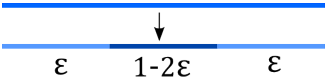
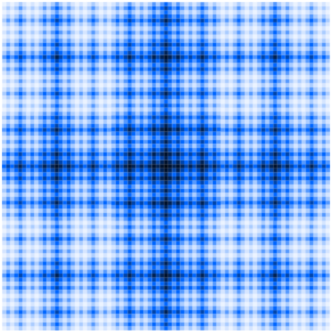
*There exist a Radon measure μ on $\mathbb{R}^2$ with $\operatorname{spt}\mu = \mathbb{R}^2$ such that μ is **doubling** ($\mu(B(x,2r)) \lesssim \mu(B(x,r))$), $\mu \perp \mathcal{H}^1$, and μ is 1-rectifiable.*





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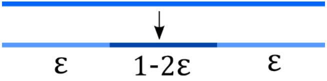

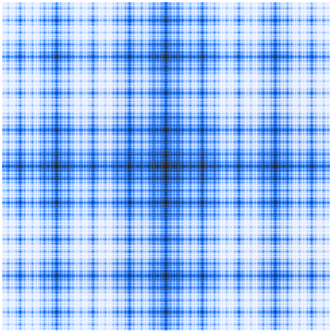

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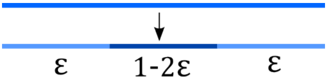
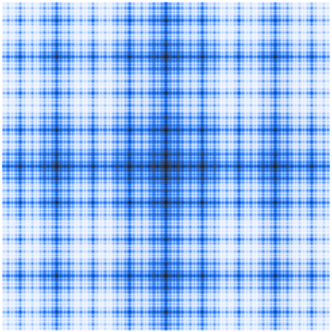




- $\lim_{r\downarrow 0} \dfrac{\mu(B(x,r))}{2r} = \infty$ μ-a.e.
- $\displaystyle\int_0^1 \left(\frac{\mu(B(x,r))}{2r}\right)^{-1} \frac{dr}{r} < \infty$ μ-a.e.

(see B-Schul 2016)

- $\mu(\Gamma) = 0$ whenever $\Gamma = f([0,1])$ and $f : [0,1] \to \mathbb{R}^2$ is bi-Lipschitz
- Nevertheless there exist Lipschitz maps $f_i : [0,1] \to \mathbb{R}^2$ such that
$$\mu\left(\mathbb{R}^2 \setminus \bigcup_{i=1}^{\infty} f_i([0,1])\right) = 0$$

Identification Problem for μ^1_{rect} and μ^1_{pu}

A Radon measure μ on $\mathbb{R}^n$ is **doubling** if $\mu(B(x,2r)) \lesssim \mu(B(x,r))$ for all $x \in \operatorname{spt}\mu$ and $r > 0$.

Solution for doubling measures with connected supports:

Theorem (Azzam-Mourgoglou 2016)

Let μ be a doubling measure on $\mathbb{R}^n$ such that $\operatorname{spt}\mu$ is connected. Then

$$\mu^1_{\text{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} > 0 \right\}$$

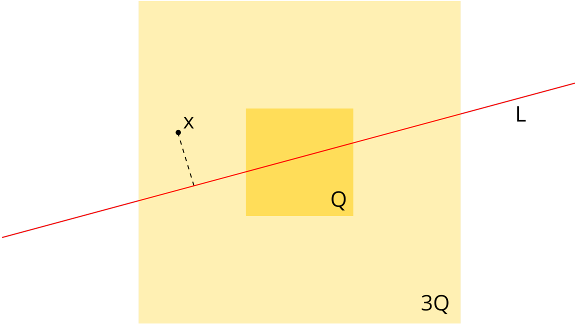
$$\mu^1_{\text{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} = 0 \right\}$$

Azzam and Mourgoglou's main result applies to doubling measures with connected supports on arbitrary metric spaces.

Non-homogeneous L^2 Jones β numbers

Let μ be a Radon measure on $\mathbb{R}^n$. For every cube Q, define $\beta_2(\mu, 3Q) = \inf_{\text{line } L} \beta_2(\mu, 3Q, L) \in [0, 1]$, where

$$\beta_2(\mu, 3Q, L)^2 = \int_{3Q} \left(\frac{\text{dist}(x, L)}{\text{diam}\, 3Q} \right)^2 \frac{d\mu(x)}{\mu(3Q)}$$



"Non-homogeneous" refers to the normalization $1/\mu(3Q)$.

Identification Problem for μ^1_{rect} and μ^1_{pu}

A Radon measure μ on $\mathbb{R}^n$ is **pointwise doubling** if

$$\limsup_{r\downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \quad \text{at } \mu\text{-a.e. } x \in \mathbb{R}^n.$$

Solution for pointwise doubling measures:

Theorem (B and Schul 2017)

Let μ be a pointwise doubling measure on $\mathbb{R}^n$. Then

$$\mu^1_{\text{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \widetilde{J}_2(\mu, x) < \infty \right\}$$

$$\mu^1_{\text{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \widetilde{J}_2(\mu, x) = \infty \right\}$$

Here $\widetilde{J}_2(\mu, x)$ is a **non-homogeneous L^2 Jones square function**:

$$\widetilde{J}_2(\mu, x) = \sum_{\substack{\text{dyadic } Q \\ \text{side } Q \leq 1}} \beta_2(\mu, 3Q)^2 \frac{\text{diam } Q}{\mu(Q)} \chi_Q(x)$$

Interpretation

Let μ be a pointwise doubling measure on $\mathbb{R}^n$. Then μ is 1-rectifiable iff

$$\widetilde{J}_2(\mu,x) = \sum_{\substack{\text{dyadic } Q\\ \text{side } Q \leq 1}} \beta_2(\mu,3Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

Extreme Behaviors

- Suppose $0 < \liminf_{r\downarrow 0} \dfrac{\mu(B(x,r))}{2r} \leq \limsup_{r\downarrow 0} \dfrac{\mu(B(x,r))}{2r} < \infty$ μ-a.e.

  Then μ is 1-rectifiable if and only if

$$\sum_{\substack{\text{dyadic } Q\\ \text{side } Q \leq 1}} \beta_2(\mu,3Q)^2 \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

  When $\mu = \mathcal{H}^1 \llcorner K$, $K \subseteq \mathbb{R}^n$ compact, this was proved by Pajot (1997)

- Suppose μ has is badly linearly approximable in the sense $\liminf_{Q\downarrow x} \beta_2(\mu,3Q) > 0$ μ-a.e. Then μ is 1-rectifiable if and only if

$$\sum_{\substack{\text{dyadic } Q\\ \text{side } Q \leq 1}} \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty \quad \mu\text{-a.e.}$$

Examples V

Theorem (Martikainen and Orponen, arXiv 2016)

For all $\varepsilon > 0$*, there exists a probability measure* μ *on* $\mathbb{R}^2$ *with* $\operatorname{spt}\mu \subseteq [0,1]^2$ *such that*

1. *pointwise non-doubling:* $\limsup_{r\downarrow 0} \dfrac{\mu(B(x,2r))}{\mu(B(x,r))} = \infty$ μ*-a.e.*
2. *vanishing lower density:* $\liminf_{r\downarrow 0} \dfrac{\mu(B(x,r))}{2r} = 0$ μ*-a.e.*
3. *uniformly bounded square function:*

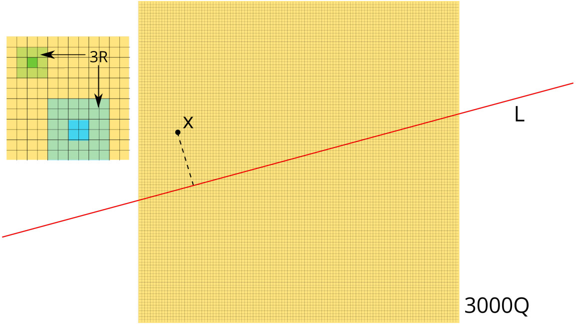
$$\widetilde{J}_2(\mu,x) = \sum_{\substack{\text{dyadic } Q \\ \text{side } Q \leq 1}} \beta_2(\mu,3Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) \leq \varepsilon \quad \textit{for all } x \in \operatorname{spt}\mu$$

Interpretation:

- vanishing lower density implies that μ is purely 1-unrectifiable
- cannot hope to characterize rectifiability of a Radon measure using only non-homogeneous square function $\widetilde{J}_2(\mu,x)$.

Anisotropic L^2 Jones β numbers

Given dyadic cube Q in $\mathbb{R}^n$, $\Delta^*(Q)$ denotes a subdivision of $1600\sqrt{n}Q$ into overlapping dyadic cubes R of same / previous (larger) generation as Q.



For every Radon measure μ on $\mathbb{R}^n$ and every dyadic cube Q, we define $\beta_2^*(\mu, Q)^2 = \inf_{\text{line } L} \max_{R \in \Delta^*(Q)} \beta_2(\mu, 3R, L)^2 m_{3R} \in [0,1]$, where

$$\beta_2(\mu, 3R, L)^2 m_{3R} = \int_{3R} \left(\frac{\text{dist}(x, L)}{\text{diam}\, 3R}\right)^2 \min\left(1, \frac{\mu(3R)}{\text{diam}\, 3R}\right) \frac{d\mu(x)}{\mu(3R)}$$

Identification Problem for μ^1_{rect} and μ^1_{pu}

Solution for Radon measures:

Theorem (B and Schul 2017)

Let μ be a Radon measure on $\mathbb{R}^n$. Then

$$\mu^1_{\mathrm{rect}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} > 0 \text{ and } J_2^*(\mu,x) < \infty \right\}$$

$$\mu^1_{\mathrm{pu}} = \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} = 0 \text{ or } J_2^*(\mu,x) = \infty \right\}$$

Here $J_2^*(\mu,x)$ is an **anisotropic L^2 Jones square function**:

$$J_2^*(\mu,x) = \sum_{\substack{\text{dyadic } Q \\ \text{side } Q \leq 1}} \beta_2^*(\mu,Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x)$$

Main Takeaway

Corollary

A Radon measure μ on $\mathbb{R}^n$ is 1-rectifiable if and only if at μ-a.e. x, $\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} > 0$ *and*

$$J_2^*(\mu,x) = \sum_{\substack{\text{dyadic } Q \\ \text{side } Q \leq 1}} \beta_2^*(\mu,Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) < \infty$$

A Radon measure μ on $\mathbb{R}^n$ is purely 1-rectifiable if and only if at μ-a.e. x, $\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{2r} = 0$ *or*

$$J_2^*(\mu,x) = \sum_{\substack{\text{dyadic } Q \\ \text{side } Q \leq 1}} \beta_2^*(\mu,Q)^2 \frac{\operatorname{diam} Q}{\mu(Q)} \chi_Q(x) = \infty$$

Takeaway:

To understand **geometric properties of non-doubling measures** (such as rectifiability) using multiscale analysis, it may be convenient or necessary to incorporate **anisotropic normalizations**.

Proof Ingredient: Drawing Rectifiable Curves

Theorem (B and Schul 2017)

Let $n \geq 2$, let $C^\star > 1$, let $x_0 \in \mathbb{R}^n$, and let $r_0 > 0$. Let $(V_k)_{k=0}^{\infty}$ be a sequence of nonempty finite subsets of $B(x_0, C^\star r_0)$ such that

1. *distinct points $v, v' \in V_k$ are uniformly separated: $|v - v'| \geq 2^{-k} r_0$;*
2. *for all $v_k \in V_k$, there exists $v_{k+1} \in V_{k+1}$ such that $|v_{k+1} - v_k| < C^\star 2^{-k} r_0$; and,*
3. *for all $v_k \in V_k$ ($k \geq 1$), there exists $v_{k-1} \in V_{k-1}$ such that $|v_{k-1} - v_k| < C^\star 2^{-k} r_0$.*

Suppose that for all $k \geq 1$ and for all $v \in V_k$ we are given a straight line $\ell_{k,v}$ in $\mathbb{R}^n$ and a number $\alpha_{k,v} \geq 0$ such that

$$\sup_{x \in (V_{k-1} \cup V_k) \cap B(v, 65 C^\star 2^{-k} r_0)} \operatorname{dist}(x, \ell_{k,v}) \leq \alpha_{k,v} 2^{-k} r_0 \tag{1}$$

and

$$\sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^2 2^{-k} r_0 < \infty. \tag{2}$$

Then the sets V_k converge in the Hausdorff metric to a compact set $V \subseteq \overline{B(x_0, C^\star r_0)}$ and there exists a compact, connected set $\Gamma \subseteq \overline{B(x_0, C^\star r_0)}$ such that $\Gamma \supseteq V$ and

$$\mathcal{H}^1(\Gamma) \lesssim_{n, C^\star} r_0 + \sum_{k=1}^{\infty} \sum_{v \in V_k} \alpha_{k,v}^2 2^{-k} r_0. \tag{3}$$

- This is a flexible criterion for drawing a rectifiable curve through the leaves of a tree; extends P. Jones' Traveling Salesman construction (Inventiones 1990), which required $V_{k+1} \supseteq V_k$
- Our write-up separates relatively simple description of the curve from the intricate length estimates

Sample of Higher Dimensional Results I

A Radon measure μ on $\mathbb{R}^n$ is ***m*-rectifiable** if $\mu(\mathbb{R}^n \setminus \bigcup \Gamma_i) = 0$ for some sequence of images Γ_i of Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$.

That is, a Radon measure is m-rectifiable provided it is carried by Lipschitz images of m-cubes.

Theorem (Preiss 1987)

Assume $\limsup_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$ *μ-a.e. (or equivalently, $\mu \ll \mathcal{H}^m$)*

Then μ is m-rectifiable if and only if $0 < \lim_{r\downarrow 0} \frac{\mu(B(x,r))}{r^m} < \infty$.

Preiss introduced tangent measures and studied global geometry of m-uniform measures in $\mathbb{R}^n$.

New examples of 3-uniform measures in $\mathbb{R}^n$ have been announced by Nimer on the arXiv in August 2016 !!!!!

Sample of Higher Dimensional Results II

A Radon measure μ on $\mathbb{R}^n$ is ***m*-rectifiable** if $\mu(\mathbb{R}^n \setminus \bigcup \Gamma_i) = 0$ for some sequence of images Γ_i of Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$.

Theorem (Azzam-Tolsa + Tolsa 2015)

Assume $0 < \limsup_{r \downarrow 0} \dfrac{\mu(B(x,r))}{r^m} < \infty$ *μ-a.e. Then μ is m-rectifiable if and only if the homogeneous L^2 Jones function*

$$J_2(\mu, x) = \int_0^1 \beta_2^{(m)}(\mu, B(x,r))^2 \frac{\mu(B(x,r))}{r^m} \frac{dr}{r} < \infty \quad \mu\text{-a.e.}$$

One ingredient in Azzam-Tolsa's proof is David and Toro's version of the Reifenberg algorithm for sets with holes.

Edelen-Naber-Valtorta announced an extension of Azzam-Tolsa on the arXiv in December 2016.

Understanding higher-dim rectifiability is hard

A Radon measure μ on $\mathbb{R}^n$ is m-**rectifiable** if $\mu(\mathbb{R}^n \setminus \bigcup \Gamma_i) = 0$ for some sequence of images Γ_i of Lipschitz maps $f_i : [0,1]^m \to \mathbb{R}^n$.

- This definition is due to Federer (1947).
- When $m \geq 2$, it is not clear if Lipschitz images of $[0,1]^m$ is the "correct" family $\mathcal{N}$ of m-dimensional sets.
- The catastrophe: If $f\colon [0,1]^m \to \mathbb{R}^n$ is Lipschitz, then $\Gamma = f([0,1]^m)$ is connected, compact, and $\mathcal{H}^m(\Gamma) < \infty$. But the converse is false when $m \geq 2$!

Open Problem: *Find additional metric, geometric, and/or topological conditions which ensure that a compact, connected set $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^2(K) < \infty$ is contained in the image of a Lipschitz map $f\colon [0,1]^2 \to \mathbb{R}^n$.*

Related developments

- **Azzam and Schul**, The Analyst's Traveling Salesman Theorem for sets of dimension larger than one, arXiv 2016.
- Novel definition of higher-dimensional beta numbers of sets using the Choquet integral
- Results include a characterization of subsets of Reifenberg vanishing bi-Lipschitz surfaces that is similar to Jones' TST.

- **K. Rajala**, Uniformization of two-dimensional metric surfaces, Inventiones 2016
- Gives an intrinsic characterization of metric spaces with locally finite $\mathcal{H}^2$ measure that are quasiconformally equivalent to $\mathbb{R}^2$
- Does not immediately extend to higher dimensions.

Current Project (w/ Vellis): Non-integral Dimensions

For each $s \in [1, n]$, let $\mathcal{N}_s$ denote all **$(1/s)$-Hölder curves** in $\mathbb{R}^n$, i.e. all images Γ of $(1/s)$-Hölder continuous maps $f\colon [0,1] \to \mathbb{R}^n$.

Decomposition: Every Radon measure μ on $\mathbb{R}^n$ can be uniquely written as $\mu = \mu_{\mathcal{N}_s} + \mu_{\mathcal{N}_s^\perp}$, where

- $\mu_{\mathcal{N}_s}$ is carried by $(1/s)$-Hölder curves
- $\mu_{\mathcal{N}_s^\perp}$ is singular to $(1/s)$-Hölder curves

Notes

- Every measure μ on $\mathbb{R}^n$ is carried by $(1/n)$-Hölder curves (space-filling curves).
- A measure μ is carried by 1-Hölder curves iff μ is 1-rectifiable.
- If μ is m-rectifiable, then μ is carried by $(1/m)$-Hölder curves.
- Martín and Mattila (1988,1993,2000) studied this concept for measures μ of the form $\mu = \mathcal{H}^s \llcorner E$, where $0 < \mathcal{H}^s(E) < \infty$

Measures with extreme lower densities

Theorem (B-Vellis, in preparation)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $s \in (1, n)$. Then the measure

$$\underline{\mu}^s_0 := \mu \llcorner \left\{ x \in \mathbb{R}^n : \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = 0 \right\}$$

is singular to $(1/s)$-Hölder curves, i.e. $\underline{\mu}^s_0(\Gamma) = 0$ whenever Γ is a $(1/s)$-Hölder curve; and, the measure

$$\underline{\mu}^s_\infty := \mu \llcorner \left\{ x \in \mathbb{R}^n : \int_0^1 \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty \text{ and } \limsup_{r \downarrow 0} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \right\}$$

is carried by $\mathcal{H}^s$ null sets of $(1/s)$-Hölder curves, i.e. there exist $(1/s)$-Hölder curves and Borel sets $N_i \subseteq \Gamma_i$ with $\mathcal{H}^s(N_i) = 0$ such that $\underline{\mu}^s_\infty(\mathbb{R}^n \setminus \bigcup_{i=1}^\infty N_i) = 0$.

- The condition $\int_0^1 \frac{r^s}{\mu(B(x,r))} \frac{dr}{r} < \infty$ implies $\lim_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s} = \infty$.
- The theorem is also true when $s = 1$ by B-Schul (2015, 2016).

Further results

Corollary (B-Vellis, in preparation)

Let μ be a Radon measure on $\mathbb{R}^n$, let $s \in [1, n)$, and let $t \in [0, s)$. Then the measure

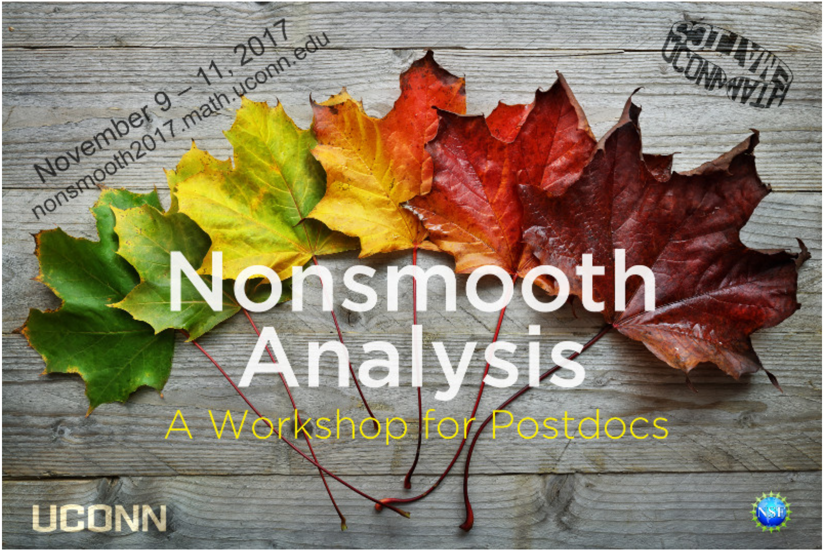
$$\mu_+^t := \mu \llcorner \left\{ x \in \mathbb{R}^n : 0 < \liminf_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} \leq \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^t} < \infty \right\}$$

is carried by $\mathcal{H}^s$ null sets of $(1/s)$-Hölder curves.

Theorem (B-Vellis, in preparation)

Let μ be a Radon measure on $\mathbb{R}^n$ and let $t \in [0, 1)$. Then the measure μ_+^t is carried by $\mathcal{H}^1$ null sets of ***bi-Lipschitz curves****, i.e. there exist bi-Lipschitz curves Γ_i and Borel sets $N_i \subseteq \Gamma_i$ with $\mathcal{H}^1(N_i) = 0$ s.t. $\mu_+^t(\mathbb{R}^n \setminus \bigcup_{i=1}^{\infty} N_i) = 0$.*

Martìn and Mattila (1988): If $0 < \mathcal{H}^t(E) < \infty$ for some $t \in [0, 1)$ and $\liminf_{r \downarrow 0} \frac{\mathcal{H}^s(E \cap B(x,r))}{r^s} > 0$ at $\mathcal{H}^s$-a.e. $x \in E$, then $\mathcal{H}^s \llcorner E$ is 1-rectifiable.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Thank you!