Quasispheres and Bi-Lipschitz Parameterization

Joint work with
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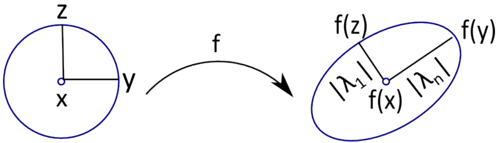
Two Measurements of Distortion

$f : \Omega \xrightarrow{\sim} \Omega'$ a homeomorphism of domains in $\mathbb{R}^n$, $f \in W^{1,n}(\Omega)$



If $Df(x)$ exists and has eigenvalues $|\lambda_1(x)| \le \cdots \le |\lambda_n(x)|$, then

$$K_f(x) = \max\left(\frac{|\lambda_n(x)|^n}{|\lambda_1(x)|\cdots|\lambda_n(x)|}, \frac{|\lambda_1(x)|\cdots|\lambda_n(x)|}{|\lambda_1(x)|^n}\right)$$

The **maximal dilatation** (local distortion)

$$K_f(\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} K_f(x).$$

The **(weak) quasisymmetry constant** (global distortion)

$$H_f(\Omega) = \max\left\{\frac{|f(y)-f(x)|}{|f(z)-f(x)|} : x, y, z \in \Omega, \frac{|y-x|}{|z-x|} \le 1\right\}$$

Local Distortion versus Global Distortion

$f : \Omega \xrightarrow{\sim} \Omega'$ a homeomorphism of domains in $\mathbb{R}^n$, $f \in W^{1,n}(\Omega)$

For all $n \geq 2$ and all domains $\Omega \subset \mathbb{R}^n$,

$$K_f(\Omega) \leq H_f(\Omega)^{n-1}$$

When $n \geq 2$ and $\Omega = \mathbb{R}^n$,

$$H_f(\mathbb{R}^n) - 1 \leq \Phi_n(K_f(\mathbb{R}^n) - 1), \quad \Phi_n : [0,\infty) \xrightarrow{\sim} [0,\infty)$$

- Precise formula for $\Phi_2(t)$ was determined by Agard 1965.
- $\Phi_n(0) = 0$ $(n \geq 3)$ was proved by Vuorinen 1989.

When $n \geq 2$, $\Omega = \mathbb{R}^n$ and K is near 1,

$$\Phi_2(K-1) \leq C_2(K-1) \quad (n=2)$$

$$\Phi_n(K-1) \leq C_n(K-1)\log\left(\frac{1}{K-1}\right) \quad (n \geq 3)$$

- Estimate $(n \geq 3)$ by Seittenranta 1996 (cf. Prause 2007)
  It is not known if the logarithm term is necessary.

Quasispheres

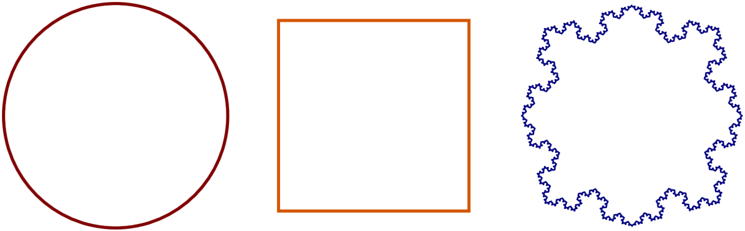
A map $f : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ is K-**quasiconformal** if

- f is a homeomorphism, • $f \in W^{1,n}(\mathbb{R}^n)$, • $K_f(\mathbb{R}^n) \leq K$.

A **quasisphere** $f(S^{n-1})$ is image of S^{n-1} under global QC map.

(A **quasicircle** $f(S^1)$ is usual name for a quasisphere in the plane.)

Examples



Question: What is the relationship between the dilatation K_f of f and the geometry of the quasisphere $f(S^{n-1})$?

Dimension of Quasispheres

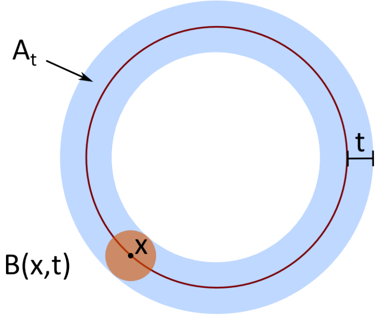
A quasisphere $f(S^{n-1})$ in $\mathbb{R}^n$ has Hausdorff dimension $d_f = \dim f(S^{n-1})$ where $n-1 \le d_f < n$.

- If $K_f(\mathbb{R}^n) = 1$, then f is Möbius. Hence $d_f = n-1$.
- Exist $f(S^{n-1})$ with d_f arbitrarily close to n. (Bishop 1999)
- $d_f \le n - 1 + c_n(K_f(\mathbb{R}^n) - 1)$ (Mattila and Vuorinen 1990)
- $d_f \le n - 1 + c_n(K_f(\mathbb{R}^n) - 1)^2 \left(\log \dfrac{1}{K_f(\mathbb{R}^n) - 1}\right)^2$
  (Prause 2007): Idea! Exploit quasisymmetry H_f of f.
- "Astala's conjecture": In the plane ($n = 2$), $d_f \le 1 + k_f^2$ where
  $k_f = (K_f(\mathbb{R}^2) - 1)/(K_f(\mathbb{R}^2) + 1)$. (Smirnov 2010)
- Refined to $\mathcal{H}^{1+k_f^2}(f(S^{n-1})) < \infty$
  (Prause, Tolsa, Uriarte-Teuro 2012)

Asymptotically Conformal Quasispheres

Natural to ask: To what degree is the geometry of $f(S^{n-1})$ determined by the dilatation of f in a neighborhood S^{n-1}?

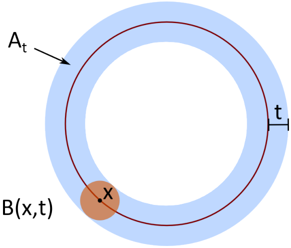
Let A_t be annular neighborhood of S^{n-1} of size t.



$f(S^{n-1})$ is **asymptotically conformal** if $K_f(A_t) \to 1$ as $t \to 0$.

- If $K_f(\mathbb{R}^n) = 1$, then $d_f = n - 1$ and $\mathcal{H}^{n-1}(f(S^{n-1})) < \infty$
- If $f(S^{n-1})$ is asymptotically conformal, then $d_f = n - 1$
- There exist asymptotical conformal $f(S^{n-1})$ such that $\mathcal{H}^{n-1}(f(S^{n-1})) = \infty$. (e.g. "flat snowflakes")
- The problem is $K_f(A_t)$ can converge to 1 very slowly!

Rectifiability



Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be quasiconformal and assume that

$$\int_0^{t_0} \Psi(K_f(A_t)-1)\frac{dt}{t} < \infty.$$

Theorem (Anderson, Becker, Lesley 1988)

If $\Psi(t) = t^2$, $n = 2$ and $f|_{B(0,1)}$ is conformal, then $\mathcal{H}^1(f(S^1)) < \infty$.

Theorem (Mattila and Vuorinen 1990)

If $\Psi(t) = t$, then $f|_{S^{n-1}}$ is Lipschitz. Hence $\mathcal{H}^{n-1}(f(S^{n-1})) < \infty$.

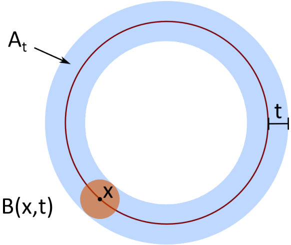
Theorem (Reshetnyak 1994)

If $\Psi(t) = t$, then $f|_{S^{n-1}}$ is a C^1 embedded submanifold of $\mathbb{R}^n$.

Rectifiability (New Result!)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiconformal and assume that

$$\int_0^{t_0} \Psi(K_f(A_t) - 1)\frac{dt}{t} < \infty.$$



Theorem (B., Gill, Rohde, Toro 2012)

If $\Psi(t) = \left(t \log t^{-1}\right)^2$, *then* $\mathcal{H}^{n-1}(f(S^{n-1})) < \infty$. *Moreover:*

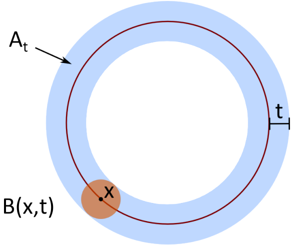
$f(S^{n-1})$ *admits local* $(1+\delta)$*-bi-Lipschitz parameterizations of* $\mathbb{R}^{n-1}$ *for every choice of* $\delta > 0$.

- This hypothesis includes the case $\Psi(t) = t^{2-\varepsilon}$, for any $\varepsilon > 0$
- The exponent 2 is the best possible, cannot replace with $2+\varepsilon$.
- The conclusion is about $f(S^{n-1})$, not about $f|_{S^{n-1}}$.
- The conclusion is weaker than saying $f(S^{n-1})$ is locally C^1

"True Version" of Theorem: Quasisymmetry Version

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiconformal and assume that

$$\int_0^{t_0} \sup_{x \in S^{n-1}} \Upsilon(H_f(B(x,t)) - 1) \frac{dt}{t} < \infty.$$



Theorem (B., Gill, Rohde, Toro 2012)

If $\Upsilon(t) = t^2$, then $\mathcal{H}^{n-1}(f(S^{n-1})) < \infty$. Moreover:

$f(S^{n-1})$ admits local $(1+\delta)$-bi-Lipschitz parameterizations of $\mathbb{R}^{n-1}$ for every choice of $\delta > 0$.

- Proof is an observation of (Prause 2007) plus a theorem on existence of bi-Lipschitz parameterizations (Toro 1995).
- Recall that $H_f(\mathbb{R}^n) - 1 \le C_n(K_f(\mathbb{R}^n) - 1) \log\left(\frac{1}{K_f(\mathbb{R}^n)-1}\right)$.
- To derive "dilatation" version of the theorem from "quasisymmetry" version , need to localize this estimate.

Dilatation Dini Condition

$[K_f(A_t)-1]^2 * [\log 1/[\ldots]]^2$

→

Quasisymmetry Dini Condition

$\sup_x [H_f(B(x,t))-1]^2$

→

Local Flatness Dini Condition

$\sup_x \theta_{f(S^{n-1})}(x,t)^2$

→

Bi-Lipschitz Parameterization

$f(S^{n-1})$ locally bi-Lipschitz equiv to R^{n-1}

Localize Known Global Estimates (BGRT 2012)

Modulus Estimates

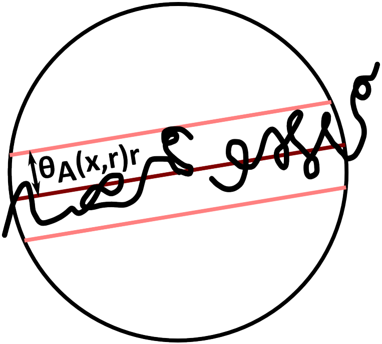
Prause's Lemma

Use Local Hölder Continuity Estimates

Toro's Theorem

Reifenberg's Topological Disk Theorem

Local Flatness of a Set



Jones β-number:

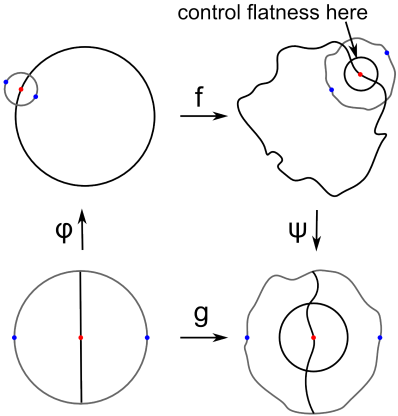
$$\beta_A(x,r) = \frac{1}{r} \inf_{L \in G(n,n-1)} \sup_{y \in A \cap B(x,r)} \operatorname{dist}(y, (x+L) \cap B(x,r))$$

Local Flatness:

$$\theta_A(x,r) = \frac{1}{r} \inf_{L \in G(n,n-1)} \operatorname{HD}[A \cap B(x,r), (x+L) \cap B(x,r)]$$

$0 \le \beta_A(x,r) \le \theta_A(x,r) \le 1$ — information when β or θ is small

Quasisymmetry Controls Local Flatness



Lemma (Prause 2007, BGRT)

If $g : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$, $g(\pm e_1) = \pm e_1$, then $\theta_{g(e_1^\perp)}(0, \frac{1}{2})$
$\leq 20[H_f(B_2) - 1]$

How does this transfer to estimates on a quasisphere?

$$\theta_{f(T_x)}(f(x), \tfrac{1}{4} f(e_r^+) - f(e_r^-)) \leq 20[H_f(B(x, 2r)) - 1]$$

where $x \in S^{n-1}$ (red point),

T_x tangent plane,

$e_r^\pm = x \pm r\vec{n}_x$ (blue points)

Future Directions

1. In joint project with Jonas Azzam and Tatiana Toro, we are looking for conditions on H_f which guarantee that $f(S^{n-1})$ is uniformly rectifiable.

2. Converse? If $f(S^{n-1})$ is a rectifiable quasisphere, must $f(S^{n-1}) = g(S^{n-1})$ for some quasiconformal map $g : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ such that H_g satisfies a square Dini condition?

3. Determine if it is possible to remove the logarithm term from the estimate $H_f(\mathbb{R}^n) - 1 \leq C_n(K_f(\mathbb{R}^n) - 1)\log\frac{1}{K_f(\mathbb{R}^n)-1}$. This is called the linear dilatation problem.