Harmonic Measure in Space

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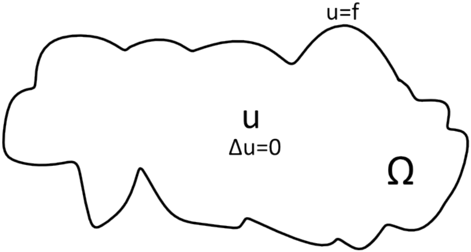
Dirichlet Problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.



Dirichlet Problem

$$(D) \quad \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

$$\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \cdots + \partial_{x_n x_n}$$

$\exists!$ family of probability measures $\{\omega^X\}_{X \in \Omega}$ on the boundary $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

$$u(X) = \int_{\partial\Omega} f(Q) d\omega^X(Q) \qquad \text{solves (D)}$$

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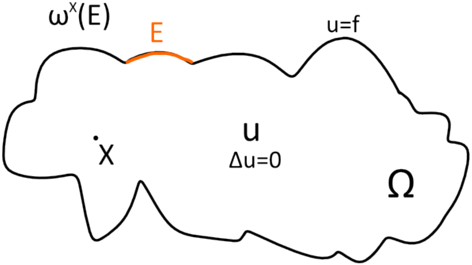

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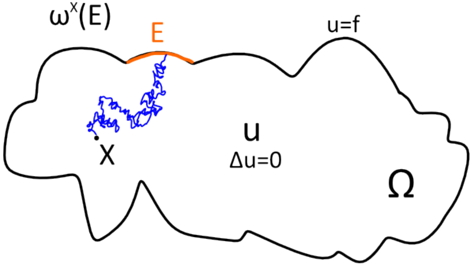

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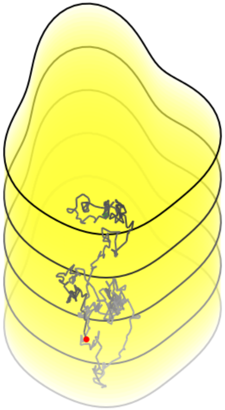
Brownian Motion Demonstrations

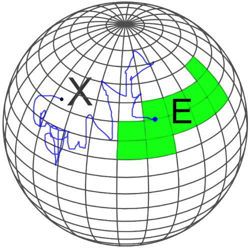


Examples of Harmonic Measure

- If $\Omega = \mathbb{B}^n$ unit ball and $X \in \Omega$, then

$$\omega^X(E) = \int_E \frac{1-|X|^2}{|X-Y|^n} \frac{d\sigma(Y)}{\sigma_{n-1}} \quad E \subset S^{n-1}$$

where $\sigma_{n-1} = \sigma(S^{n-1})$.



- If $\Omega \subset \mathbb{R}^n$ is bounded domain of class C^1, then $\exists\ K(X,Y) : \Omega \times \partial\Omega \to \mathbb{R}$ such that

$$\omega^X(E) = \int_E K(X,Y) d\sigma(Y) \quad E \subset \partial\Omega.$$

- We call the Radon-Nikodym derivative $\dfrac{d\omega^X}{d\sigma} = K(X,\cdot)$ the **Poisson kernel** of Ω.

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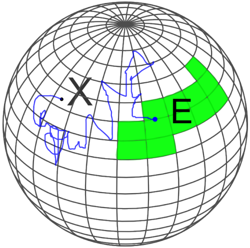

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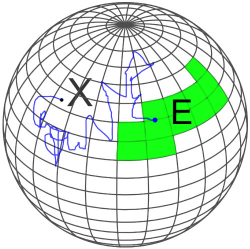

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More Examples of Harmonic Measure

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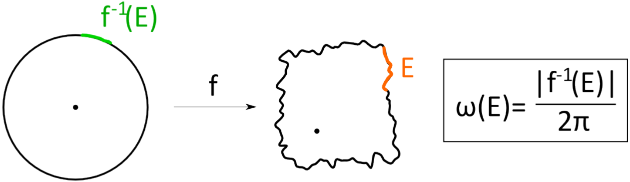

- If $\Omega = A(0, r, R) \subset \mathbb{R}^n$ is an annular region, then the harmonic measure of "the inner shell" $S(0, r)$ is

$$\omega^X(S(0,r)) = \begin{cases} \dfrac{\log R - \log |X|}{\log R - \log r} & \text{if } n = 2 \\ \\ \dfrac{|X|^{2-n} - R^{2-n}}{r^{2-n} - R^{2-n}} & \text{if } n \geq 3 \end{cases}$$

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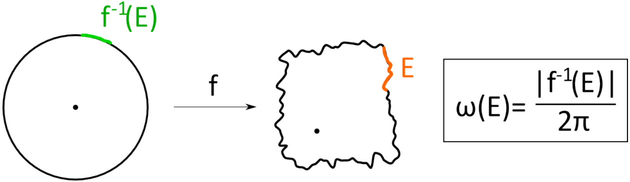

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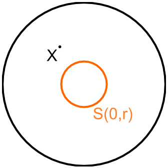
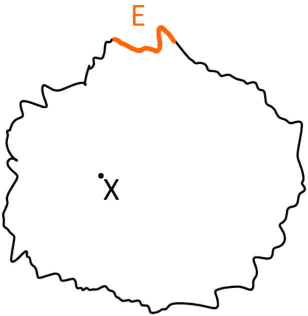


Harmonic Measure at Different Poles



For every Borel set $E \subset \partial\Omega$, $X \mapsto \omega^X(E)$ is a nonnegative harmonic function in Ω

Harmonic measures ω^{X_1} and ω^{X_2} at different poles are

mutually absolutely continuous $(\omega^{X_1}(E) = 0 \Leftrightarrow \omega^{X_2}(E) = 0)$.

"Drop the pole" to get **"the harmonic measure"** ω of Ω

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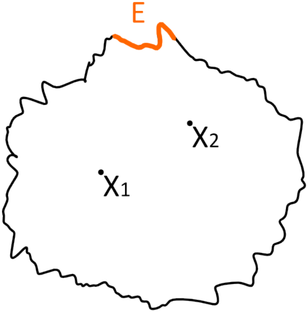

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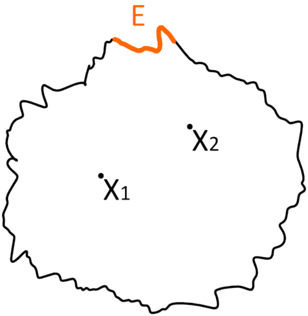

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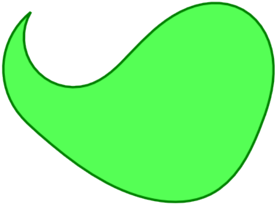

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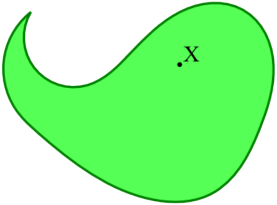

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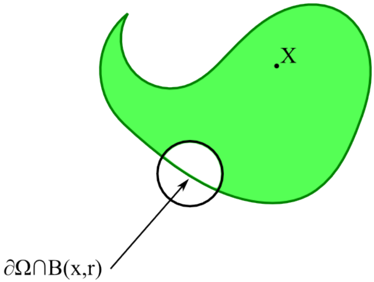

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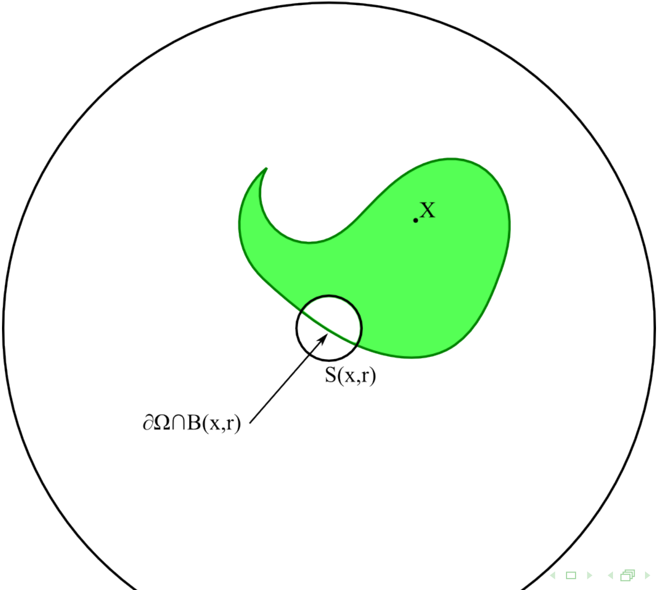

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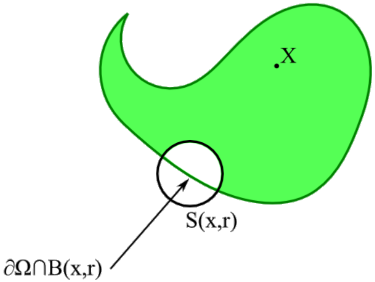
$$\omega^X(\partial\Omega \cap B(x,r)) \leq \frac{r^{n-2}}{\mathrm{dist}(X, \partial\Omega)}$$



Dimension of Harmonic Measure

Let $\dim E$ denote the Hausdorff dimension of a set $E \subset \mathbb{R}^n$.

The (upper) Hausdorff dimension of harmonic measure is smallest dimension of a set with full harmonic measure:

$$\dim \omega = \inf\{\dim E : \omega(\mathbb{R}^n \setminus E) = 0\}.$$

On a general domain $\Omega \subset \mathbb{R}^n$ we have the bounds:

$$n - 2 \leq \dim \omega \leq n - b_n \quad \text{for some } b_n > 0$$

- The lower bound is easy (previous slide)
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Harmonic Measure in the Plane

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  These are now called Wolff snowflakes.
- (Lewis, Verchota and Vogel 2005) There exist complementary domains $\Omega^+ = \Omega \subset \mathbb{R}^n$ and $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ such that
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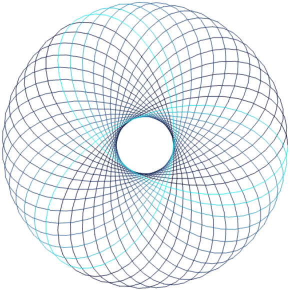
- (Badger) "F. and M. Riesz Theorem for NTA domains": If $\Omega \subset \mathbb{R}^n$ NTA and if $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then $\mathcal{H}^{n-1}|_{\partial\Omega} \ll \omega$. Thus $\dim \omega = n - 1$.

Open Problems

- Compute Bourgain's constant b_n for $n \geq 3$.
- Does $b_n \to 0$ as $n \to \infty$ or is $\inf_n b_n > 0$?
- Let $\Omega \subset \mathbb{R}^n$ be an NTA domain such that $\mathcal{H}^{n-1}(\partial\Omega) < \infty$. Show $\omega \ll \mathcal{H}^{n-1}|_{\partial\Omega}$ (or find a counterexample).

Regularity of Harmonic Measure



Recall if $\Omega \subset \mathbb{R}^n$ is bounded domain and $\partial\Omega$ is at least C^1, then

$$\omega^X(E) = \int_E \frac{d\omega^X}{d\sigma}(Y)d\sigma(Y) \quad E \subset \partial\Omega.$$

Regularity of the Boundary $\Longrightarrow$ Regularity of the Poisson Kernel:

- $\partial\Omega$ is $C^\infty \Rightarrow \log \frac{d\omega}{d\sigma} \in C^\infty(\partial\Omega)$
- $\partial\Omega$ is C^k and $k \geq 2 \Rightarrow \log \frac{d\omega}{d\sigma} \in C^{k-1}(\partial\Omega)$
- $\partial\Omega$ is $C^{1,\alpha}$ and $\alpha > 0 \Rightarrow \log \frac{d\omega}{d\sigma} \in C^{0,\alpha}(\partial\Omega)$ (Kellogg 1929)
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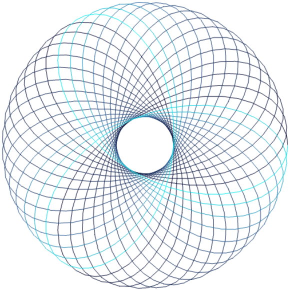

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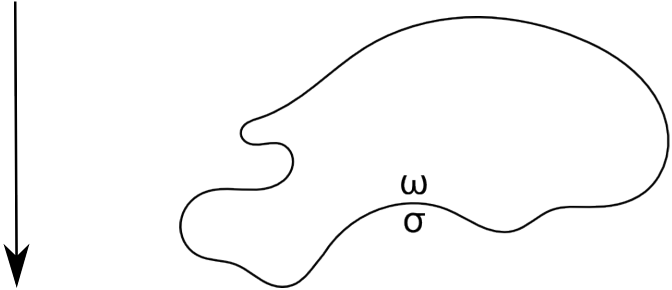
Free Boundary Problem 1



Let $\Omega \subset \mathbb{R}^n$ be a domain of locally finite perimeter, with harmonic measure ω and surface measure $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$.

If the **Poisson kernel** $\dfrac{d\omega}{d\sigma}$ *is sufficiently regular,*

then how regular is the boundary $\partial\Omega$?

FBP 1 Results

- (Kinderlehrer and Nirenberg 1977) Let $\Omega \subset \mathbb{R}^n$ be of class C^1.
  1. $\log \frac{d\omega}{d\sigma} \in C^{1+m,\alpha}$ for $m \geq 0$, $\alpha \in (0,1) \implies \partial\Omega$ is $C^{2+m,\alpha}$.
  2. $\log \frac{d\omega}{d\sigma} \in C^\infty \implies \partial\Omega$ is C^∞
  3. $\log \frac{d\omega}{d\sigma}$ is real analytic $\implies \partial\Omega$ is real analytic.
- (Alt and Caffarelli 1981) Assume $\Omega \subset \mathbb{R}^n$ satisfies necessary "weak conditions" (that includes C^1 as a special case). Then: $\log \frac{d\omega}{d\sigma} \in C^{0,\alpha}$ for $\alpha > 0 \implies \partial\Omega$ is $C^{1,\beta}$, $\beta = \beta(\alpha) > 0$.
- (Jerison 1987) In Alt and Caffarelli's Theorem, $\beta = \alpha$.
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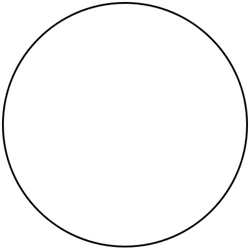
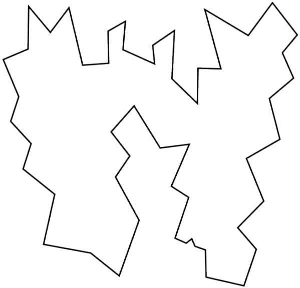
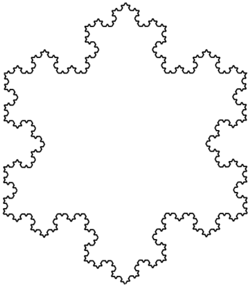
Examples of NTA Domains



Smooth Domains



Lipschitz Domains



Quasispheres

(e.g. snowflake)

Question: How should we measure regularity of harmonic measure on domains which do not have surface measure?

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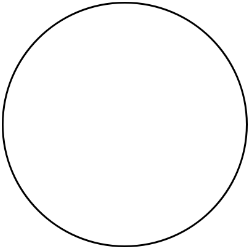

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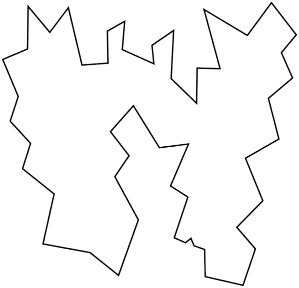

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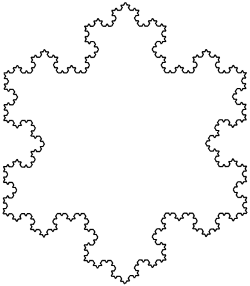

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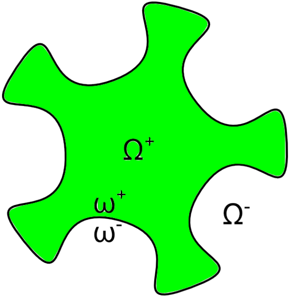
Free Boundary Problem 2



$\Omega \subset \mathbb{R}^n$ is a **2-sided domain** if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial\Omega^+ = \partial\Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with interior harmonic measure ω^+ and exterior harmonic measure ω^-

If the **two-sided kernel** $\dfrac{d\omega^-}{d\omega^+}$ *is sufficiently regular,*

then how regular is the boundary $\partial\Omega$?

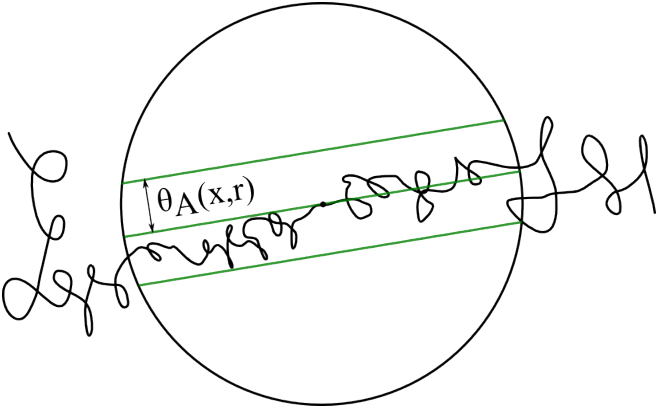
An Unexpected Example

$\log \frac{d\omega^-}{d\omega^+}$ is smooth $\not\Rightarrow$ $\partial\Omega$ is smooth

Figure: The zero set of the harmonic polynomial
$h(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y) - 10xyz$

$\Omega^\pm = \{h^\pm > 0\}$ is a 2-sided domain, $\omega^+ = \omega^-$ (pole at infinity),
$\log \frac{d\omega^-}{d\omega^+} \equiv 0$ but $\partial\Omega^\pm = \{h = 0\}$ is not smooth at the origin.

Local Flatness



- Always have $\theta_A(x,r) \leq 1$ — only get information if θ is small
- $A \subset \mathbb{R}^n$ is δ-**Reifenberg flat**, if $\theta_A(x,r) \leq \delta \; \forall x \in A$, $r \leq r_0$
- $A \subset \mathbb{R}^n$ is **Reifenberg flat with vanishing constant** if A is δ-Reifenberg flat for all $\delta > 0$

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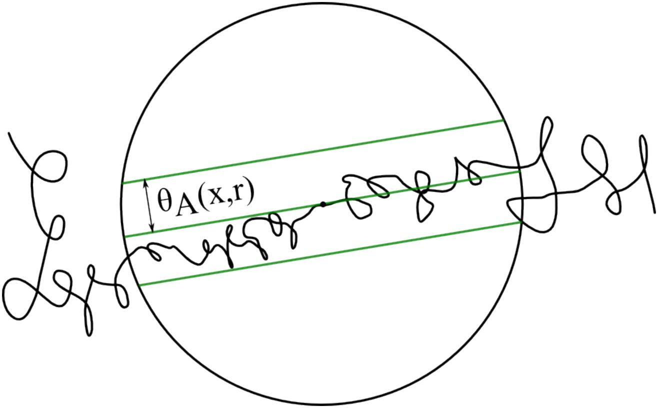

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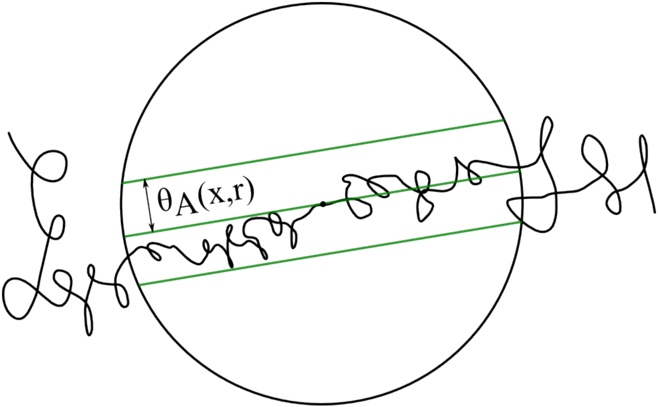

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Blow-up of a Set (= Zooming In on a Set)

Let $A \subset \mathbb{R}^n$ be a closed set and let $x \in A$.
We say B is a **blow-up** of A at $x \in A$ if $\exists$ radii $r_i \downarrow 0$ so that

$$\frac{A - x}{r_i} \to B \quad \text{in Hausdorff distance, uniformly on compact sets.}$$

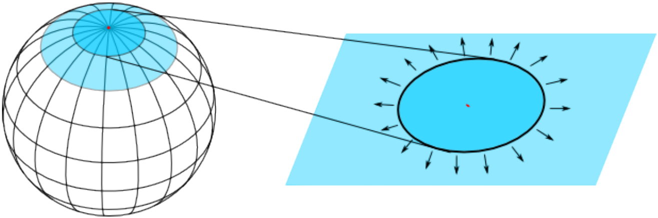

Figure: A blow-up of S^{n-1} at the north pole is the plane $\mathbb{R}^{n-1} \times \{0\}$

Structure Theorem for FBP 2

Theorem (Badger) Assume $\Omega \subset \mathbb{R}^n$ is a 2-sided NTA domain, $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in C^0(\partial\Omega)$.

There exists $d \geq 1$ (depending on the NTA constants) such that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.

1. Every **blow-up** of $\partial\Omega$ about a point $Q \in \Gamma_k$ is the **zero set** $h^{-1}(0)$ of a homogeneous harmonic polynomial h of degree k which separates $\mathbb{R}^n$ into two components.
2. The "flat points" Γ_1 is an open subset of $\partial\Omega$ and is (locally) Reifenberg flat with vanishing constant.
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Flat Points $Q \in \Gamma_1$

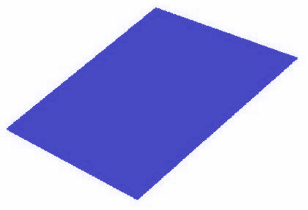

Figure: $h^{-1}(0)$ when h homogeneous harmonic polynomial of degree 1, i.e. a plane through the origin

Remarks

- Γ_1 has full measure: $\omega^{\pm}(\partial\Omega \setminus \Gamma_1) = 0$.
- Γ_1 is open in $\partial\Omega$ and Reifenberg flat with vanishing constant. Thus $\dim \Gamma_1 = n - 1$.
- At $Q \in \Gamma_1$ one can see different planes as $\lim_{i\to\infty} \frac{\partial\Omega - Q}{r_i}$ along different sequences of scales $r_i \downarrow 0$.
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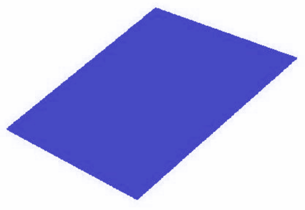

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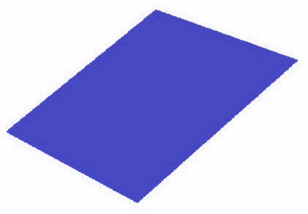

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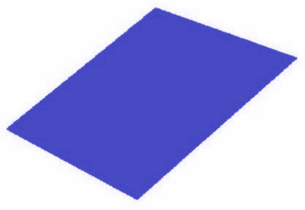

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Singularities $Q \in \Gamma_2 \cup \cdots \cup \Gamma_d$

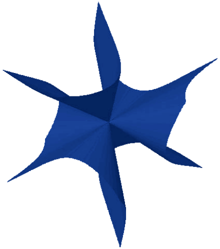

Figure: $h^{-1}(0)$ where $h(x,y,z) = x^2(y-z) + y^2(z-x) + z^2(x-y) - xyz$ is an example blow-up of $\partial\Omega$ about $Q \in \Gamma_3$, $\Omega \subset \mathbb{R}^3$.

Remarks

- $\Gamma_2 \cup \cdots \cup \Gamma_d \subset \partial\Omega$ is closed and $\omega^{\pm}(\Gamma_2 \cup \cdots \cup \Gamma_d) = 0$.
- Examples shows that $\dim(\Gamma_2 \cup \cdots \cup \Gamma_d) = n-3$ is possible. Upper bound is unknown.
- In dimension $n=3$, $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \Gamma_5 \cup \cdots \cup \Gamma_{2k+1}$.
  (by a Result about Spherical Harmonics by Lewy 1977)

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- Examples shows that $\dim(\Gamma_2 \cup \cdots \cup \Gamma_d) = n - 3$ is possible. Upper bound is unknown.
- In dimension $n = 3$, $\partial\Omega = \Gamma_1 \cup \Gamma_3 \cup \Gamma_5 \cup \cdots \cup \Gamma_{2k+1}$.
  (by a Result about Spherical Harmonics by Lewy 1977)

Singularities $Q \in \Gamma_2 \cup \cdots \cup \Gamma_d$

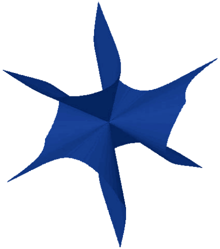

Figure: $h^{-1}(0)$ where $h(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y) - xyz$ is an example blow-up of $\partial\Omega$ about $Q \in \Gamma_3$, $\Omega \subset \mathbb{R}^3$.

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Ingredients in the Proof

1. FBP 2 was studied by Kenig and Toro (2006) who showed that blow-ups of $\partial\Omega$ are zero sets of harmonic polynomials.
   - We show that only zero sets of homogeneous harmonic polynomials appear as blow-ups.
   - We show the degree of polynomials appearing in blow-ups is unique at every $Q \in \partial\Omega$. Hence $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.
   - We study topology and size of the sets Γ_k.
2. To classify geometric blow-ups of the boundary, we study measure-theoretic blow-ups of ω (tangent measures).
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(Conjecture: $\dim \Gamma_k \leq n - 3$.)

3. If $\log \frac{d\omega^-}{d\omega^+}$ is $C^{0,\alpha}$, then is Γ_1 locally the $C^{1,\alpha}$ image of a hyperplane?

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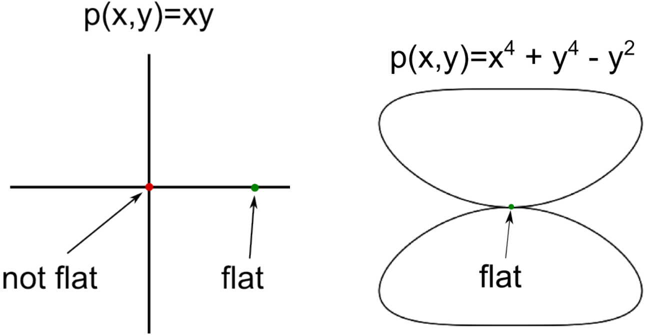
Flat Points

If $p : \mathbb{R}^n \to \mathbb{R}$ is a polynomial (with real coefficients), then $\Sigma_p = \{X \in \mathbb{R}^n : p(X) = 0\}$ is its zero set.

We say $X \in \Sigma_p$ is a **flat point** if $\lim_{r \downarrow 0} \theta_{\Sigma_p}(X, r) = 0$.



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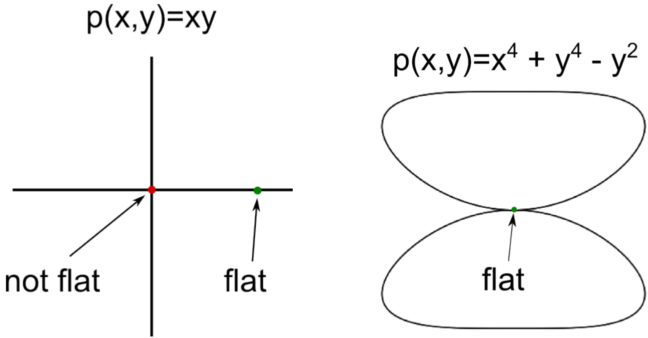

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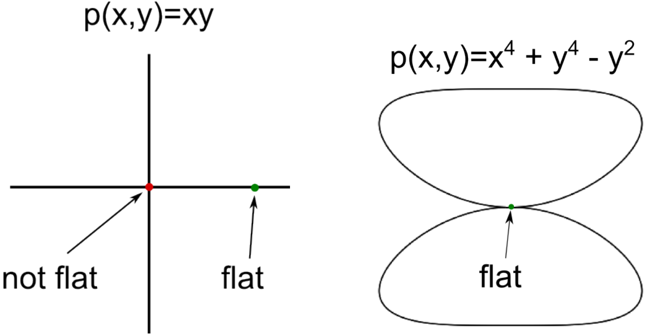

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Theorem (Badger) For each $n \geq 2$ and $d \geq 2$, there exists $\delta_{n,d} > 0$ with the following property:

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Corollary (Badger) If $p : \mathbb{R}^n \to \mathbb{R}$ is a harmonic polynomial, then $\{X \in \Sigma_p : X \text{ is flat}\} = \{X \in \Sigma_p : Dp(X) \neq 0\}$.

Open Problem Find all polynomials $p : \mathbb{R}^n \to \mathbb{R}$ whose zero set Σ_p has the feature that its flat points and regular points coincide.

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