Harmonic Measure from Two Sides (and Tools from Geometric Measure Theory)

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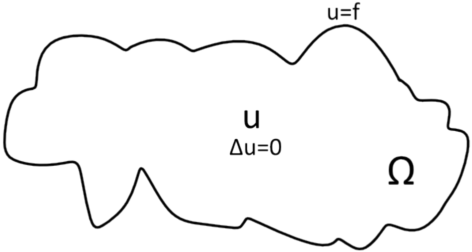
Dirichlet Problem

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain.



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$$(D) \begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega \end{cases}$$

$$\Delta = \partial_{x_1 x_1} + \partial_{x_2 x_2} + \cdots + \partial_{x_n x_n}$$

$\exists!$ family of probability measures $\{\omega^X\}_{X \in \Omega}$ on the boundary $\partial\Omega$ called **harmonic measure** of Ω with pole at $X \in \Omega$ such that

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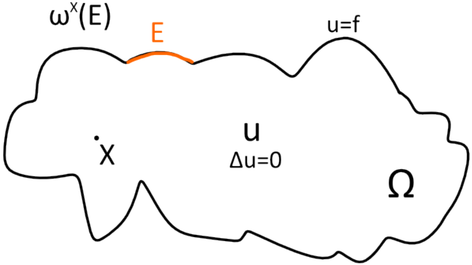

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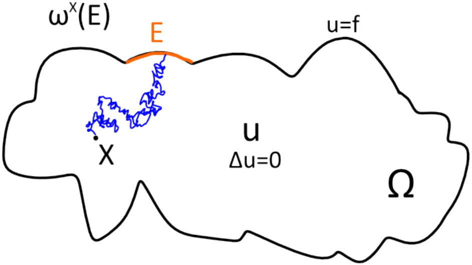

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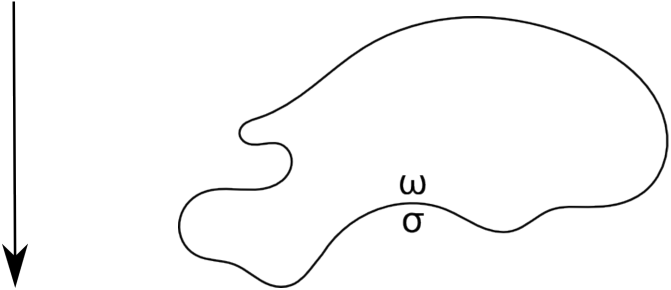
Free Boundary Problem 1



Let $\Omega \subset \mathbb{R}^n$ be a domain of locally finite perimeter, with harmonic measure ω and surface measure $\sigma = \mathcal{H}^{n-1}|_{\partial\Omega}$.

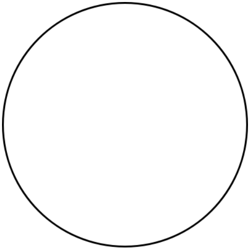
If the **Poisson kernel** $\dfrac{d\omega}{d\sigma}$ *is sufficiently regular,*

then how regular is the boundary $\partial\Omega$?

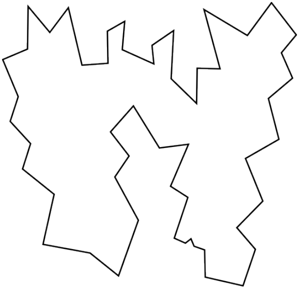
FBP 1 Results

- (Kinderlehrer and Nirenberg 1977) Let $\Omega \subset \mathbb{R}^n$ be of class C^1.
  1. $\log \frac{d\omega}{d\sigma} \in C^{1+m,\alpha}$ for $m \geq 0$, $\alpha \in (0,1) \implies \partial\Omega$ is $C^{2+m,\alpha}$.
  2. $\log \frac{d\omega}{d\sigma} \in C^\infty \implies \partial\Omega$ is C^∞
  3. $\log \frac{d\omega}{d\sigma}$ is real analytic $\implies \partial\Omega$ is real analytic.
- (Alt and Caffarelli 1981) Assume $\Omega \subset \mathbb{R}^n$ satisfies necessary "weak conditions" (that includes C^1 as a special case). Then: $\log \frac{d\omega}{d\sigma} \in C^{0,\alpha}$ for $\alpha > 0 \implies \partial\Omega$ is $C^{1,\beta}$, $\beta = \beta(\alpha) > 0$.
- (Jerison 1987) In Alt and Caffarelli's Theorem, $\beta = \alpha$.
- (Jerison 1987) $\log \frac{d\omega}{d\sigma} \in C^0 \implies \partial\Omega$ is VMO_1.
- (Kenig and Toro 2003) Studied FBP 1 with $\log \frac{d\omega}{d\sigma} \in \mathrm{VMO}$.

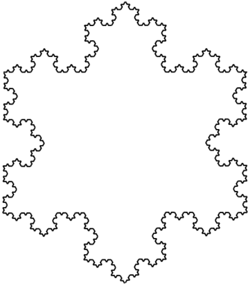
Examples of NTA Domains



Smooth Domains



Lipschitz Domains

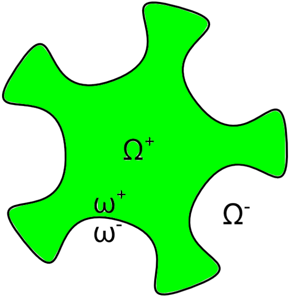


Quasispheres

(e.g. snowflake)

Question: How should we measure regularity of harmonic measure on domains which do not have surface measure?

Free Boundary Problem 2



$\Omega \subset \mathbb{R}^n$ is a **2-sided domain** if:

1. $\Omega^+ = \Omega$ is open and connected
2. $\Omega^- = \mathbb{R}^n \setminus \overline{\Omega}$ is open and connected
3. $\partial\Omega^+ = \partial\Omega^-$

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided domain, equipped with interior harmonic measure ω^+ and exterior harmonic measure ω^-

If the **two-sided kernel** $\dfrac{d\omega^-}{d\omega^+}$ *is sufficiently regular,*

then how regular is the boundary $\partial\Omega$?

An Unexpected Example

$\log \frac{d\omega^-}{d\omega^+}$ is smooth **does not imply** $\partial\Omega$ is smooth

Figure: The zero set of the harmonic polynomial
$h(x, y, z) = x^2(y - z) + y^2(z - x) + z^2(x - y) - 10xyz$

$\Omega^\pm = \{h^\pm > 0\}$ is a 2-sided domain, $\omega^+ = \omega^-$ (pole at infinity),
$\log \frac{d\omega^-}{d\omega^+} \equiv 0$ but $\partial\Omega^\pm = \{h = 0\}$ is not smooth at the origin.

Structure Theorem for FBP 2

Theorem (B) Assume $\Omega \subset \mathbb{R}^n$ is a 2-sided NTA domain, $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+)$ or $\log \frac{d\omega^-}{d\omega^+} \in C^0(\partial\Omega)$.

There exists $d \geq 1$ (depending on the NTA constants) such that $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.

1. Every **blow-up** of $\partial\Omega$ about a point $Q \in \Gamma_k$ is the **zero set** $h^{-1}(0)$ of a homogeneous harmonic polynomial h of degree k which separates $\mathbb{R}^n$ into two components.
2. The "flat points" Γ_1 is a dense open subset of $\partial\Omega$ with Hausdorff dimension $n-1$.
3. The "singularities" $\Gamma_2 \cup \cdots \cup \Gamma_d$ have harmonic measure zero.

Ingredients in the Proof

1. FBP 2 was studied by Kenig and Toro (2006) who showed that blow-ups of $\partial\Omega$ are zero sets of harmonic polynomials.
   - We show that only zero sets of homogeneous harmonic polynomials appear as blow-ups.
   - We show the degree of polynomials appearing in blow-ups is unique at every $Q \in \partial\Omega$. Hence $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d$.
   - We study topology and size of the sets Γ_k.
2. To classify geometric blow-ups of the boundary, we study measure-theoretic blow-ups of $\omega^{\pm}$ (tangent measures).
3. To show Γ_1 is open, we study local flatness properties of the zero sets of harmonic polynomials.

Polynomial Harmonic Measures

$h : \mathbb{R}^n \to \mathbb{R}$ be a polynomial, $\Delta h = 0$

$\Omega^+ = \{X : h(X) > 0\}$, $\Omega^- = \{X : h(X) < 0\}$
(i.e. $h^\pm$ is the Green function for $\Omega^\pm$)

The harmonic measure ω_h associated to h is the harmonic measure of $\Omega^\pm$ with pole at infinity; i.e., for all $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{h^{-1}(0)} \varphi d\omega_h = -\int_{\partial\Omega^\pm} \varphi \frac{\partial h^\pm}{\partial \nu} d\sigma = \int_{\Omega^\pm} h^\pm \Delta\varphi$$

Two Collections of Measures Associated to Polynomials

$\mathcal{P}_d = \{\omega_h : h \text{ harmonic polynomial of degree} \leq d\}$

$\mathcal{F}_k = \{\omega_h : h \text{ homogenous harmonic polynomial of degree} = k\}$

Blow-ups of the Boundary $\longleftrightarrow$ Tangent Measures of ω

Let $\Omega \subset \mathbb{R}^n$ be a 2-sided NTA domain, let $Q \in \partial\Omega$ and let $r_i \downarrow 0$.

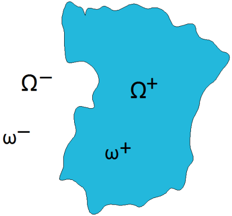
Theorem: (KT) There is subsequence of r_i (which we relabel) and an unbounded 2-sided NTA domain Ω_∞ such that

- Blow-ups of Boundary at Q Converge:
$$\partial\Omega_i = \frac{\partial\Omega - Q}{r_i} \to \partial\Omega_\infty \text{ in Hausdorff metric}$$
- Blow-ups of Harmonic Measure at Q Converge:
$$\omega_i^\pm(E) = \frac{\omega^\pm(Q + r_i E)}{\omega^\pm(B(Q, r_i))} \text{ satisfy } \omega_i^\pm \rightharpoonup \omega_\infty^\pm$$
where $\omega_\infty^\pm$ is the harmonic measure of $\Omega_\infty^\pm$ with pole at infinity.

Each blow-up $\omega_\infty^\pm$ is called a tangent measure of $\omega^\pm$ at Q.

Tangent Measures of $\omega^{\pm}$ when $\omega^+ \ll \omega^- \ll \omega^+$

Theorem (Kenig and Toro)
If $\omega^+ \ll \omega^- \ll \omega^+$ and $\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+)$,
then $\mathrm{Tan}(\omega^{\pm}, Q) \subset \mathcal{P}_d$.



Goal: Show $\mathrm{Tan}(\omega^{\pm}, Q) \subset \mathcal{F}_k$
for some $1 \leq k = k(Q) \leq d$.

$$\mathcal{P}_d = \{\omega_h : h \text{ harmonic polynomial of degree} \leq d\}$$
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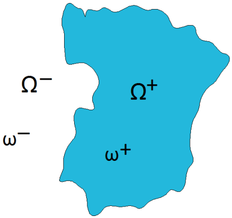

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Cones of Measures

A collection $\mathcal{M}$ of non-zero Radon measures is a d-cone if it preserved under scaling and dilation of $\mathbb{R}^n$:

1. If $\nu \in \mathcal{M}$ and $c > 0$, then $c\nu \in \mathcal{M}$.
2. If $\nu \in \mathcal{M}$ and $r > 0$, then $T_{0,r\sharp}\nu \in \mathcal{M}$. $[T_{0,r}(y) = y/r]$

Examples

- Tangent Measures: $\mathrm{Tan}(\mu, x)$
- Polynomial Harmonic Measures: $\mathcal{P}_d$ and $\mathcal{F}_k$

Size of a Measure and Distance to a Cone

- Let ψ be Radon measure on $\mathbb{R}^n$. The "size" of ψ on $B(0,r)$ is
$$F_r(\psi) = \int_0^r \psi(B(0,s))ds.$$
- Let ψ be a Radon measure on $\mathbb{R}^n$ and $\mathcal{M}$ a d-cone. There is a "distance" $d_r(\psi, \mathcal{M})$ from ψ to $\mathcal{M}$ on $B(0,r)$ compatible with weak convergence of measures.

Connectedness of Tangent Measures

Let $\mathcal{F}$ and $\mathcal{M}$ be d-cones such that $\mathcal{F} \subset \mathcal{M}$. Assume that:

- $\mathcal{F}$ and $\mathcal{M}$ have compact bases ($\{\psi : F_1(\psi) = 1\}$),
- (Property P) There exists $\epsilon_0 > 0$ such that whenever $\mu \in \mathcal{M}$ and $d_r(\mu, \mathcal{F}) < \epsilon_0$ for all $r \geq r_0$ then $\mu \in \mathcal{F}$.

Theorem

If $\mathrm{Tan}(\nu, x) \subset \mathcal{M}$ *and* $\mathrm{Tan}(\nu, x) \cap \mathcal{F} \neq \emptyset$, *then* $\mathrm{Tan}(\nu, x) \subset \mathcal{F}$.

Key Point: (Under technical hypotheses) If one tangent measure at a point belongs to $\mathcal{F}$ then all tangent measures belong to $\mathcal{F}$.

First proved in [P], the theorem was stated in this form in [KPT].

- Preiss used the theorem to show Radon measures in $\mathbb{R}^n$ with positive and finite m-density almost everywhere are m-rectifiable.
- Kenig, Preiss and Toro used the theorem to compute Hausdorff dimension of harmonic measure when $\omega^+ \ll \omega^- \ll \omega^+$.

Checking the Hypotheses: Rate of Doubling

- If $\omega \in \mathcal{F}_k$, then $\omega(B(0,r)) = cr^{n+k-2}$ where c depends on n, k and $\|h\|_{L^1(S^{n-1})}$. Thus $\mathcal{F}_k$ is uniformly doubling: if $\omega \in \mathcal{F}_k$ then
$$\frac{\omega(B(0,2r))}{\omega(B(0,r))} = 2^{n+k-2} \quad \text{for all } r > 0$$
independent of the associated polynomial h.
Lemma: $\mathcal{F}_k$ has compact basis for all $k \geq 1$.

- If $\omega \in \mathcal{P}_d$ is associated to a polynomial of degree $j \leq d$ (not necessarily homogeneous), then for all $\tau > 1$
$$\frac{\omega(B(0,\tau r))}{\omega(B(0,r))} \sim \tau^{n+j-2} \quad \text{as } r \to \infty.$$
Theorem: The comparison constant depends only on n and j!
Corollary: If $d_r(\omega, \mathcal{F}_k) < \varepsilon_0(n,d)\ \forall\, r \geq r_0(\omega)$, then $k = j$.

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Polynomial Blow-ups are Homogeneous

Theorem (B)

Let Ω be a 2-sided NTA domain. If $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{P}_d$, then $\mathrm{Tan}(\omega^\pm, Q) \subset \mathcal{F}_k$ for some $1 \leq k \leq d$.

Steps in the Proof

1. Since $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{P}_d$, there is a smallest degree $k \leq d$ such that $\mathrm{Tan}(\omega^+, Q) \cap \mathcal{P}_k \neq \emptyset$. Show that $\mathrm{Tan}(\omega^+, Q) \cap \mathcal{P}_k \subset \mathcal{F}_k$.
2. Let $\mathcal{F} = \mathcal{F}_k$ and $\mathcal{M} = \mathrm{Tan}(\omega^+, Q) \cup \mathcal{F}_k$. By the previous slide the hypotheses of the connectedness theorem are satisfied. Therefore, $\mathrm{Tan}(\omega^+, Q) \subset \mathcal{F}_k$.

Open Questions

$$\log \frac{d\omega^-}{d\omega^+} \in VMO(d\omega^+) \Rightarrow \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_d.$$

1. Find an upper bound on dimension of the "singularities" $\Gamma_2 \cup \cdots \cup \Gamma_d$. (Conjecture: $\dim_H \leq n-3$)

2. (Higher Regularity) For example, if $\log \frac{d\omega^-}{d\omega^+} \in C^{0,\alpha}$, then at $Q \in \Gamma_k$ is $\partial\Omega$ locally the $C^{1,\alpha}$ image of the zero set of a harmonic polynomial of degree k?

3. (Rectifiability) Does $\Gamma_1 = G \cup N$ where G is $(n-1)$-rectifiable and $\omega^{\pm}(N) = 0$?
   - The answer is yes if one assumes that $\partial\Omega$ has locally finite perimeter (Kenig-Preiss-Toro, B).

4. Find other applications of the connectedness of tangent measures.

REFERENCES

M. Badger, *Harmonic polynomials and tangent measures of harmonic measure*, Rev. Mat. Iberoam. **27** (2011), no. 3, 841–870. arXiv:0910.2591

M. Badger, *Flat points in zero sets of harmonic polynomials and harmonic measure from two sides*, preprint. arXiv:1109.1427

APPENDIX

Distance from Measure to a Cone

Let $\mathcal{L}(r) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \geq 0, \, \mathrm{Lip} f \leq 1, \, \mathrm{spt}\, f \subset B(0, r)\}$.

If μ and ν are two Radon measures in $\mathbb{R}^n$ and $r > 0$, we set

$$F_r(\mu, \nu) = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : f \in \mathcal{L}(r) \right\}.$$

When $\nu = 0$,

$$F_r(\mu, 0) = \int_0^r \mu(B(0, s)) ds =: F_r(\mu).$$

Note that $\mu_i \rightharpoonup \mu$ if and only if $\lim_{i \to \infty} F_r(\mu_i, \mu) = 0$ for all $r > 0$.

If ψ is a Radon measure and $\mathcal{M}$ is a d-cone, we define a scaled version of F_r as follows:

$$d_r(\psi, \mathcal{M}) = \inf \left\{ F_r \left(\frac{\psi}{F_r(\psi)}, \mu \right) : \mu \in \mathcal{M} \text{ and } F_r(\mu) = 1 \right\},$$

i.e., normalize ψ so $F_r(\psi) = 1$ & then take distance to $\mathcal{M}$ on B_r.

Homogeneous Harmonic Polynomials – "Big Piece" Lemma

Let $h : \mathbb{R}^n \to \mathbb{R}$ be homogenous harmonic polynomial of degree k.

Key Lemma (B): There is a constant $\ell_{n,k} > 0$ with the following property. For all $t \in (0, 1)$,

$$\mathcal{H}^{n-1}\{\theta \in S^{n-1} : |h(\theta)| \geq t\|h\|_{L^\infty(S^{n-1})}\} \geq \ell_{n,k}(1-t)^{n-1}.$$

Interpretation: If h is homogeneous harmonic polynomial, then h takes big values on a big piece of the unit sphere.

Bounds for $\omega(B(0,r))$ as $r \to \infty$

Given $h : \mathbb{R}^n \to \mathbb{R}$ harmonic polynomial of degree d, $h(0) = 0$,

$$h = h_d + h_{d-1} + \cdots + h_1$$

where h_k is homogeneous harmonic polynomial of degree k.
In polar coordinates,

$$h(r\theta) = r^d h_d(\theta) + r^{d-1} h_{d-1}(\theta) + \cdots + r h_1(\theta),$$

$$\frac{dh}{dr}(r\theta) = d r^{d-1} h_d(\theta) + (d-1) r^{d-2} h_{d-1}(\theta) + \cdots + h_1(\theta).$$

Fact: Recall $\Omega^+ = \{X : h(X) > 0\}$. For all $r > 0$,

$$\omega(B(0,r)) = \int_{\partial B(0,r) \cap \Omega^+} \frac{dh^+}{dr} d\sigma$$

The $r^{d-1} h_d(\theta)$ term dominates as $r \to \infty$. Upper bounds for $\omega(B_r)$ are easy. Use "Big Piece" Lemma for lower bounds.