3D Brownian Motion and Sets of Dimension 2.99999 99999 99999

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February 2024

Joint work with **Alyssa Genschaw**
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Colloquium
The University of Tennessee, Knoxville
February 2024

Research Partially Supported by NSF DMS 2154047

Harmonic functions

A continuous function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is **harmonic** if its value $u(X)$ at any point X is the average of u in a neighborhood of X:

$$u(X) = \frac{1}{\mathrm{vol}(B_X(r))} \int_{B_X(r)} u(Y)\, dY \quad \text{for every ball } B_X(r) \subset \Omega$$

Theorem

u is harmonic if and only if u is C^2 and it satisfies the PDE $\Delta u \equiv u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = 0$ (Laplace's equation) throughout Ω

Examples

- $(n = 1)$: every harmonic function $u : (a, b) \to \mathbb{R}$ is linear, $u(x) = cx + d$ for some c and d, since $u''(x) = 0$ for all $x \in (a, b)$
- $(n = 2)$: $u(x, y) = \sin(x)e^y$ and $u(x, y) = x^3 - 3xy^2$ are examples of nonlinear harmonic functions on $\mathbb{R}^2$

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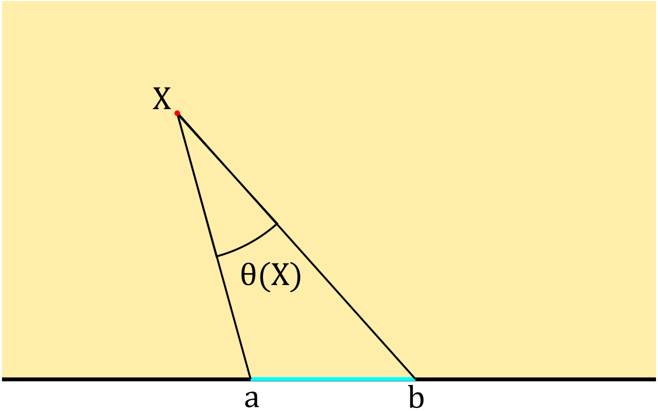
An example in the upper half plane

Let $\Omega = \{X = (x, y) : y > 0\}$ be the upper half plane.

Let (a, b) be an interval on x-axis.



The angle function $\theta(X)$ is harmonic.

Another important example: hitting probabilities

Fix an open region $\Omega \subset \mathbb{R}^n$ (picture $n = 2$ or $n = 3$). Draw a random curve B_t, $t \geq 0$ starting from some point $B_0 = X$ inside Ω

This is called **Brownian motion** started from X

Let $\tau = \tau_{\Omega^c}$ denote the first (random) time such that $B_\tau \notin \Omega$.
As $t \mapsto B_t$ is continuous, the exit location $B_\tau \in \partial\Omega$, the boundary of Ω

Fix a set $E \subset \partial\Omega$. The function $\omega^E : \Omega \to [0, 1]$ defined by $\omega^E(X) = \mathbf{Prob}(B_\tau \in E : B_0 = X)$ is called **harmonic measure** of E

"Probability that random curve started at X first hits $\partial\Omega$ in E"

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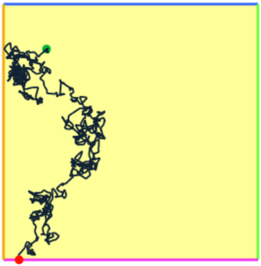
Brownian motion / harmonic measure demos

Brownian Motion Demonstrations

Matthew Badger **February 23, 2024**

[Demo 1] | [Demo 2] | [Demo 3]



Physical interpretations

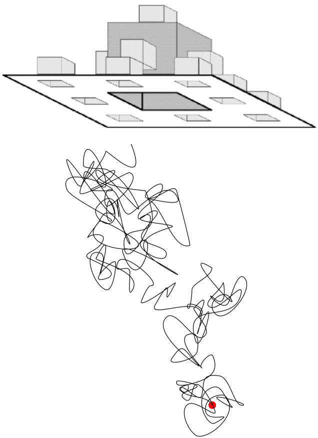
Diffusion describes the net movement of molecules from regions of higher concentration to regions of lower concentration.

Brownian motion models the trajectory of a single molecule undergoing diffusion

Examples of physical phenomena controlled by diffusion

- Heat transfer
- Electric charge on a metallic surface
- Passivation / fouling / poisoning / restricted absorbtion
- Transfer of nutrients from digestive system to blood
- Transfer of oxygen from air to blood during respiration

Passiviation of irregular surfaces (Filoche et al.)



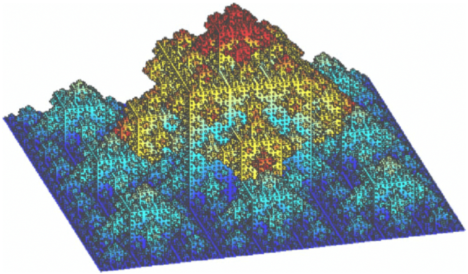


Fig. 3. Successive active regions during the passivation process of a fifth-generation prefractal surface based on the cubic Koch surface. Diffusing particles are coming from below and reach first mainly the blue region of the interface. Each color in the simulation represents a set of four successive passivated regions (dark blue corresponds to regions 1–4, light blue to 5–8, etc.). One can see that the size of the active region gradually decreases during the passivation process. At the end, only the dark-red regions on the tip are active.

Interlude: What is dimension?

3d Reconstruction of Human Lungs from CT-scans

Cantor sets

$C(\lambda)$, $0 < \lambda < 1/2$:

$$C(\lambda) = \bigcap_{k=0}^{\infty} C_k(\lambda)$$

1. At stage k: $C_k(\lambda)$ has 2^k intervals of length λ^k
2. Lebesgue measure (total length) $\mathcal{L}(C_k(\lambda)) = (2\lambda)^k, \quad 0 < 2\lambda < 1$
3. Lebesgue measure
   $\mathcal{L}(C(\lambda)) = \lim_{k\to\infty} \mathcal{L}(C_k(\lambda)) = \lim_{k\to\infty}(2\lambda)^k = 0$

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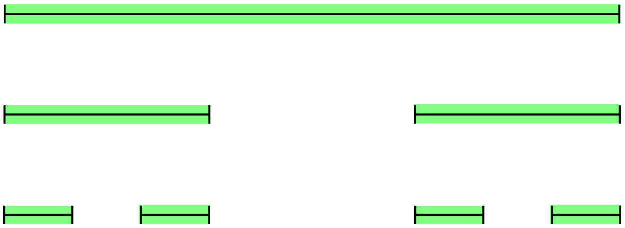

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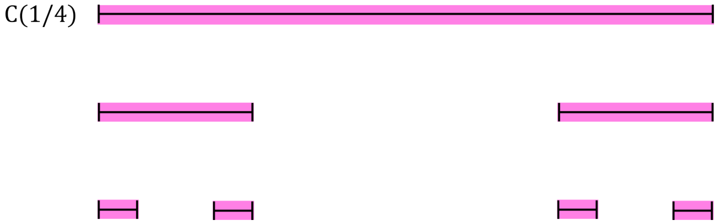

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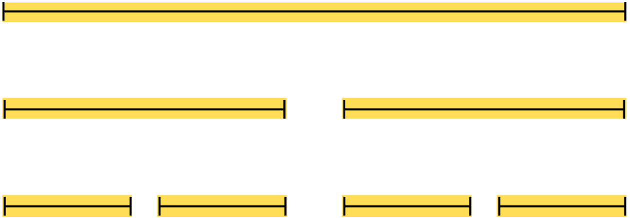
C(9/20)



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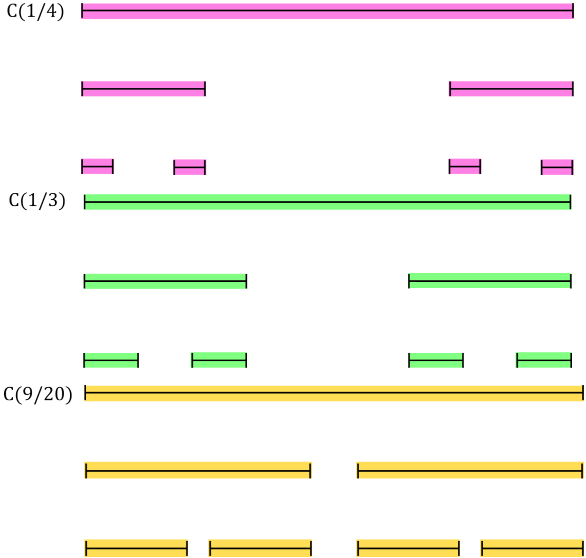
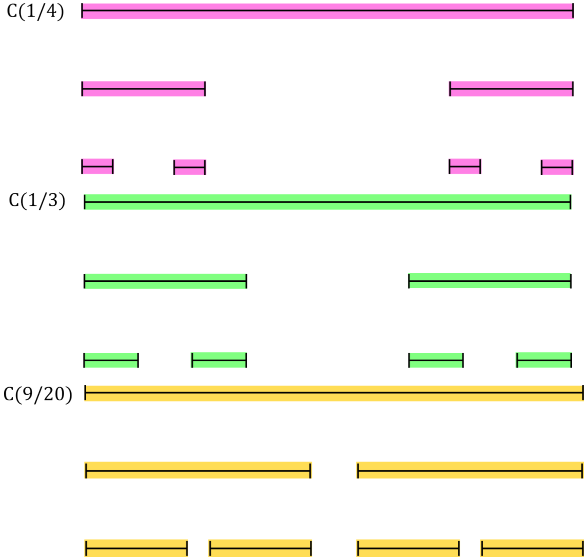
Length cannot distinguish $C(1/3)$, $C(1/4)$, $C(9/20)$



Cantor sets

But our intuition says that $C(1/4)$ "<" $C(1/3)$ "<" $C(9/20)$



s-dimensional Hausdorff measures $\mathcal{H}^s$ on $\mathbb{R}^n$

Let $A \subseteq \mathbb{R}^n$ be any set. Let $s \geq 0$ be any nonnegative real number

1. For any $\delta > 0$ cover A by cubes $Q_1, Q_2, \ldots$ of side length $\leq \delta$
2. Weight each cube in the cover by its side length to power s
3. Optimize over all such covers

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} (\text{side } Q_i)^s : A \subset \bigcup_{i=1}^{\infty} Q_i, \ \text{side } Q_i \leq \delta \right\}$$

4. Use only finer and finer covers

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

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Some examples / properties of Hausdorff measure

1. Open balls $B_{\mathbb{R}^n}(x, r)$ in $\mathbb{R}^n$ have $\mathcal{H}^n(B(x, r)) = c(n)r^n$

2. If $s < n < t$, then $\mathcal{H}^s(B_{\mathbb{R}^n}(x, r)) = \infty$ and $\mathcal{H}^t(B(_{\mathbb{R}^n}(x, r)) = 0$

3. Line segments $[a, b]$ in $\mathbb{R}^n$ have $\mathcal{H}^1([a, b]) = |b - a|$

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Some examples / properties of Hausdorff measure

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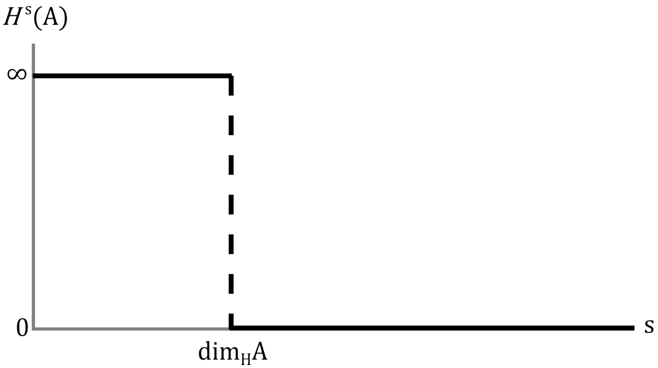
Hausdorff dimension

For any set $A \subseteq \mathbb{R}^n$, there is a unique number $d \in [0, n]$ such that

1. $\mathcal{H}^s(A) = \infty$ for all $s < d$
2. $\mathcal{H}^s(A) = 0$ for all $s > d$

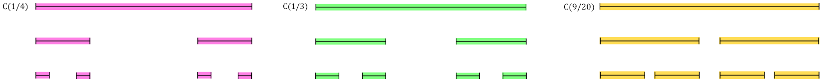


The number $d = \dim_H(A)$ where the transition happens is called the **Hausdorff dimension of** A.

Hausdorff dimension of Cantor sets

For all $\lambda \in (0, 1/2)$, the Cantor set $C(\lambda)$ has Hausdorff dimension

$$\dim_H C(\lambda) = \frac{\log(2)}{\log(1/\lambda)} \in (0, 1)$$



- $\dim_H C(1/4) = \log(2)/\log(4) = 0.5000000\ldots$
- $\dim_H C(1/3) = \log(2)/\log(3) = 0.6309292\ldots$
- $\dim_H C(9/20) = \log(2)/\log(20/9) = 0.8680532\ldots$
- $\dim_H C(\lambda) \downarrow 0$ as $\lambda \downarrow 0$
- $\dim_H C(\lambda) \uparrow 1$ as $\lambda \uparrow 1/2$

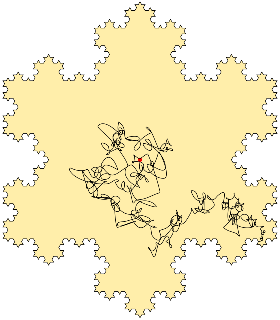
Dimension of human lungs

Estimates of "Fractal Dimension" in the Literature

Approximation of scaling laws for different measurements of lungs along branching pathways (airway length, airway diameter) from experimental data leads to:

- Nelson and Machester (1988): dimension is ≈ 2.64 and ≈ 2.76
- Nelson, West, and Golden (1990): dimension is ≈ 2.26 and ≈ 2.4
- Weibel (1991): dimension is ≈ 2.35
- Lamrini Uahabi and Atounti (2017): dimension is ≈ 2.88.

Harmonic measure on the Koch snowflake domain



Theorem

$\dim_H \partial\Omega = \log(4)/\log(3) = 1.26185\ldots$

Theorem (Kaufman and Wu 1985)

There is a subset $E \subset \partial\Omega$ with $\dim_H E < \dim_H \partial\Omega$ such that $\omega^E(X) = 1$ and $\omega^{\partial\Omega\setminus E}(X) = 0$ for every $X \in \Omega$.

We call this phenomena **dimension drop**

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There is a subset $E \subset \partial\Omega$ with $\dim_H E = 1$ such that $\omega^E(X) = 1$ and $\omega^{\partial\Omega\setminus E}(X) = 0$ for every $X \in \Omega$. That is, Brownian motion only hits the boundary of the snowflake in a subset of dimension 1.

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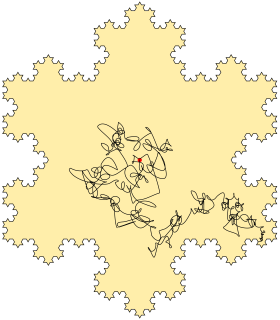

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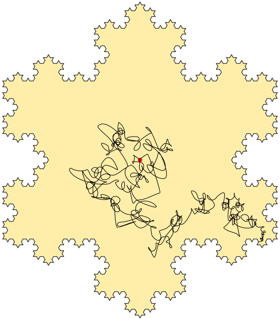

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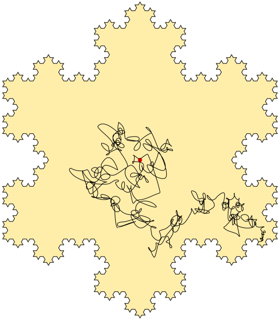

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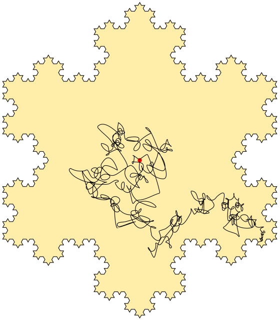

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Dimension of harmonic measure

The **dimension of harmonic measure** on a domain $\Omega \subset \mathbb{R}^n$ is the smallest dimension $\dim_H E$ of a subset $E \subset \partial\Omega$ with

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Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. The dimension of harmonic measure is 1, and moreover, any set Hausdorff dimension less than 1 has harmonic measure zero.

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No matter how large the dimension of the boundary $\partial\Omega$ is, the dimension of harmonic measure on $\Omega \subset \mathbb{R}^2$ is at most 1.

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We do not know. This question is open!

Random curves do not favor any particular direction, so the dimension of harmonic measure on a ball in $\mathbb{R}^3$ is 2. Hence the dimension of harmonic measure in $\mathbb{R}^3$ is ≥ 2.

Theorem (Wolff 1995)

The dimension of harmonic measure in $\mathbb{R}^3$ is > 2.
(This doesn't say how much greater than 2.)

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Numerical Experiment (Grebenkov et al. 2005)

The dimension of harmonic measure in $\mathbb{R}^3$ *is* ≥ 2.005

- This estimate is made by simulating Brownian motion on the fifth iteration of a cubical Koch snowflake surface.
- This bound is not yet mathematically verified.

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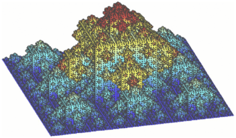

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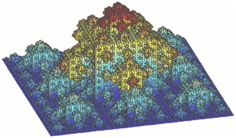

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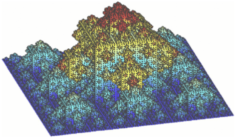

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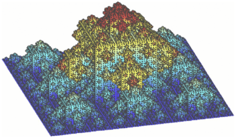

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Main idea: Bourgain's alternative

Theorem (Bourgain 1987, Badger-Genschaw 2021)

Suppose you can find constants $\rho = \rho(n) > 0$ and $\lambda = \lambda(n) > 0$ such that for every domain $\Omega \subset \mathbb{R}^n$ and every cube Q, either:

- ***Alternative 1**: At the resolution of Q, the boundary of the domain inside the cube looks at most $n - \rho$ dimensional: $\mathcal{H}^{n-\rho}_{\frac{1}{2}\,\text{side}\,Q}(\partial\Omega \cap Q) < (\text{side}\,Q)^{n-\rho}$*
- ***Alternative 2**: The boundary of the domain inside of Q is uniformly spread throughout the cube, which makes it unlikely that Brownian motion started outside the cube will first hit $\partial\Omega$ near the center of the cube. Quantify this using the constant λ.*

Then the dimension of harmonic measure in $\mathbb{R}^n$ is at most $n - \frac{\lambda\rho}{\lambda+\rho}$.

Theorem (Bourgain 1987)

For any $n \geq 3$, the constants $\rho = \rho(n) > 0$ and $\lambda = \lambda(n) > 0$ exist. Therefore, the dimension of harmonic measure in $\mathbb{R}^n$ is $< n$.

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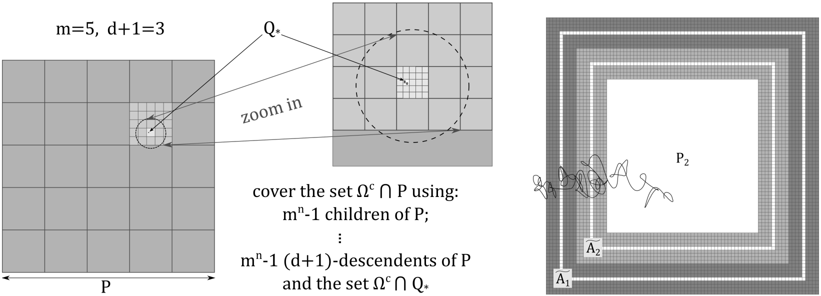
Dimension of harmonic measure in $\mathbb{R}^3$ is $< 2.99999\,99999\,99999$

Idea: Introduce a model of Bourgain's alternative in which we are able to rigorously compute admissible pairs of constants λ and ρ.



We then optimize over several discrete parameters in the the model to find the best possible bound for the dimension of harmonic measure in $\mathbb{R}^3$.

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Table A.1. Bounding Bourgain's constant for harmonic measure: $b_n \geq \lambda\rho/(\lambda+\rho) \approx \rho$ when $\rho \ll \lambda$.

n	m	η	h	d	α	γ	λ	ρ
3	5	0.0005	2	7	303.102	0.9976...	$1.488... \times 10^{-3}$	$8.020... \times 10^{-18}$
3	6	0.0008	2	6	277.560	0.9947...	$2.911... \times 10^{-3}$	$1.801... \times 10^{-17}$
3	7	0.0019	3	5	83.8178	0.9998...	$7.481... \times 10^{-5}$	$2.418... \times 10^{-16}$
3	8	0.0011	3	5	81.9976	0.9965...	$1.678... \times 10^{-3}$	$2.215... \times 10^{-17}$
3	9	0.0046	3	4	60.8979	0.9996...	$1.616... \times 10^{-4}$	$1.452... \times 10^{-15}$
3	10	0.0031	4	4	61.4480	0.9992...	$3.385... \times 10^{-4}$	$3.210... \times 10^{-16}$
3	11	0.0022	4	4	54.2657	0.9984...	$6.516... \times 10^{-4}$	$8.031... \times 10^{-17}$
3	12	0.0016	5	4	55.5835	0.9993...	$2.254... \times 10^{-4}$	$2.174... \times 10^{-17}$
3	13	0.0012	5	4	52.5339	0.9982...	$6.978... \times 10^{-4}$	$6.521... \times 10^{-18}$
3	14	0.0009	5	4	54.1918	0.9988...	$4.385... \times 10^{-4}$	$2.117... \times 10^{-18}$

Coda

There is a **large gap** between the experimental lower bound 2.005 and the rigorous upper bound $2.99999\,99999\,99999$ on the dimension of harmonic measure in $\mathbb{R}^3$.

Both bounds are far away from Bishop's conjectural value of 2.5.

The current methods are not optimal and there are **many possible directions** to get improved estimates or compute the dimension of harmonic measure in $\mathbb{R}^3$.

Theorem (Badger and Genschaw 2023)

Dimension of harmonic measure in $\mathbb{R}^n$ is $< n - 0.6n^{-2n^2}$ for $n \gg 3$.

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What about harmonic-type measures for other operators?

A first step... David and Mayboroda (2021) and Perstneva (2023) give examples of elliptic measures associated to variable coefficient operators on fractal planar domains which have **zero dimension drop**.

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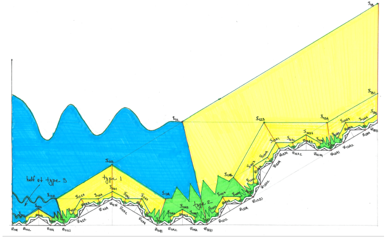

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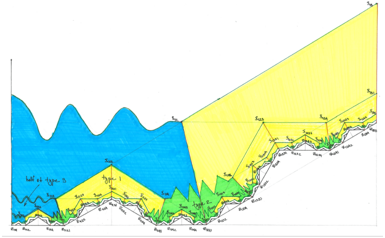

—End—


Thank you for your attention!
Glimpse of Connecticut on Tuesday last week